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A SUBNORMAL SEMIGROUP WITHOUT NORMAL EXTENSION[1]

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ABSTRACT. THEOREM 1. *There exists a subnormal semigroup with no commuting normal extension.* THEOREM 2. *There exist two commuting quasinormal operators without commuting normal extension.*

1. A bounded linear operator T on a Hilbert space $\mathcal{H}$ is called subnormal if there exists a normal operator N acting on a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that $\mathcal{H}$ is invariant under N and $N|_{\mathcal{H}} = T$. T. Ito [3] showed that commuting subnormals $T_1, \ldots, T_n$ on $\mathcal{H}$ have a commuting normal extension, i.e. there exist commuting normals $N_1, \ldots, N_n$ each defined on $\mathcal{K} \supset \mathcal{H}$ with $N_i|_{\mathcal{H}} = T_i$, if and only if an analog of the Halmos-Bram positivity condition is satisfied. Ito also showed that every continuous one-parameter semigroup of commuting subnormals on $\mathcal{H}$ can be extended to a continuous one-parameter semigroup of commuting normals on some $\mathcal{K} \supset \mathcal{H}$. Recent examples by M. B. Abrahamse [1] and A. Lubin [4] show that there exist commuting subnormals with no commuting normal extensions. In this note we give an example of a two-parameter subnormal semigroup without commuting normal extension; our example also provides two commuting quasinormal operators without commuting normal extension. Our example is presented in a simplified form suggested by Professor Chandler Davis.

2. Let $\mathcal{H}$ be the Hilbert space having orthonormal basis $\{c_0, e_n, f_n : n = 1, 2, \ldots\}$, and define U_1, U_2 on $\mathcal{H}$ by:

$$\begin{aligned} U_1(e_n) &= e_{n+1}, \\ U_1(c_0) &= e_1, \\ U_1(f_n) &= 0, \\[1ex] U_2(e_n) &= 0, \\ U_2(c_0) &= f_1, \\ U_2(f_n) &= f_{n+1}, \end{aligned}$$

i.e.

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$$U_1 = \left(\begin{array}{c|c|c} \begin{matrix}\ddots & & \\ & \ddots & \\ & & \ddots \\ & 0 & 1 \\ & & 0\end{matrix} & \begin{matrix}\cdot \\ \\ \\ \\ 1\end{matrix} & \\ \hline & 0 & \\ \hline & & \bigcirc \end{array}\right), \quad U_2 = \left(\begin{array}{c|c|c} \bigcirc & & \\ \hline & 0 & \\ \hline & \begin{matrix}1 \\ \\ \\ \\ \end{matrix} & \begin{matrix}0 & & \\ 1 & \cdot & \\ & \cdot\ \cdot & \\ & & \cdot\ \cdot \\ & & 1\end{matrix} \end{array}\right)$$

Note that each U_i is the direct sum of a unilateral shift and 0, and that $U_1U_2 = U_2U_1 = 0$. U_1 and U_2 are clearly subnormal, and are even quasinormal. (Recall that an operator A is quasinormal if $A(A^*A) = (A^*A)A$ and every quasi-normal operator is subnormal [**2**, p. 101].) Since the powers of a subnormal are subnormal, $\mathcal{S} = \{U_1^jU_2^k: j, k = 0, 1, 2, \dots\}$ is a commutative subnormal semigroup.

If there exists a commutative normal semigroup $\{N_1^jN_2^k\}$ on $\mathcal{K}$ extending $\mathcal{S}$, then $(U_1 + U_2) = (N_1 + N_2)|_{\mathcal{K}}$ is subnormal. An operator A is called hyponormal if $(A^*A - AA^*) \geqslant 0$, and all subnormal operators are hyponormal [**2**, p. 103]. A simple computation shows that for $X = U_1 + U_2$,

$$Q = X^*X - XX^* = \left(\begin{array}{c|c|c} \bigcirc & & \\ & & -1 \\ \hline & 2 & \\ \hline -1 & & \\ & & \bigcirc \end{array}\right)$$

Thus, $(Q(e_1 + f_1), (e_1 + f_1)) = -2 < 0$ so Q is not even hyponormal and therefore $\mathcal{S}$ cannot have a normal extension. Hence, we have

THEOREM 1. *There exists a commuting subnormal semigroup with no commuting normal extension.*

THEOREM 2. *There exist two commuting quasinormal operators without a commuting normal extension.*

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