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DIVISIBILITY PROPERTIES OF THE q-TANGENT NUMBERS

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ABSTRACT. The q-tangent number $T_{2n+1}(q)$ is shown to be divisible by $(1+q)(1+q^2)\cdots(1+q^n)$. Related divisibility questions are discussed.

1. Introduction. The tangent numbers T_{2n+1} are integers defined by

$$\sum_{n=0}^{\infty} \frac{T_{2n+1}x^n}{(2n+1)!} = \tan x. \tag{1.1}$$

Numerous properties of the tangent number are known; in particular [**2**, p. 259]:

$$T_{2n+1} = 4^{n+1}|G_{2n+2}|/(n+1), \tag{1.2}$$

where G_n is an integer called the Genocchi number. Thus it is clear from (1.2) that T_{2n+1} is always divisible by a high power of 2.

A natural q-analog of the tangent numbers is given by

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{T_{2n+1}(q)x^n}{(q)_{2n+1}} &= \frac{\sin_q x}{\cos_q x} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(q)_{2n+1}} \Big/ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(q)_{2n}}, \end{aligned} \tag{1.3}$$

where $(A)_n = (A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1})$; R. P. Stanley [**4**] has given a combinatorial interpretation of the polynomials $T_{2n+1}(q)$ which shows that all the coefficients are nonnegative.

One of us [**3**] has shown that $T_{2n+1}(q)$ is divisible by the cyclotomic polynomials $\phi_2(q), \phi_4(q), \ldots, \phi_{2n}(q)$ through a study of properties of Gaussian polynomials in cyclotomic fields. Our object here is to derive the following result on q-tangent numbers which is analogous to the fact that T_{2n+1} is divisible by a high power of 2:

THEOREM 1. *The polynomial* $T_{2n+1}(q)$ *is divisible by* $(1+q)(1+q^2)\cdots(1+q^n)$.

We conclude with a few comments about other divisibility properties of $T_{2n+1}(q)$ that are derivable using our method. The assertion in Theorem 1 was a conjecture made by M. P. Schützenberger at the combinatorics

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2. Proof of Theorem 1. To prove this result we require two lemmas.

LEMMA 1. *For nonnegative integers N and j, the expression*

$$\begin{bmatrix} 2N+1 \\ 2j \end{bmatrix} \frac{(1+q)(1+q^2)\cdots(1+q^j)}{(1+q^N)(1+q^{N-1})\cdots(1+q^{N-j+1})} \tag{2.1}$$

is a polynomial in q, where $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is the Gaussian polynomial

$$\begin{bmatrix} N \\ M \end{bmatrix} = \frac{(q)_N}{(q)_M(q)_{N-M}}. \tag{2.2}$$

PROOF. Obviously the expression in question is a rational function and the roots of the denominator are roots of unity. To prove Lemma 1 we need only show that each zero of the denominator appears with at least as large multiplicity in the numerator as in the denominator.

Now if ρ is a primitive kth root of unity then ρ is a *simple* root of $1 - q^M$ if and only if $k|M$. Furthermore we know a priori (due to the recurrences for Gaussian polynomials) that $\left[\begin{smallmatrix} 2N+1 \\ 2j \end{smallmatrix}\right]$ is a polynomial. Consequently for each integer l with $1 \leqslant l \leqslant 2j$, we see that l must divide at least $[2j/l]$ of the numbers $2N+1, 2N, 2N-1, \ldots, 2N-2j+2$ (otherwise this Gaussian polynomial would not be a polynomial). Now

(2.3)

$$\begin{aligned}\begin{bmatrix} 2N+1 \\ 2j \end{bmatrix} & \frac{(1+q)(1+q^2)\cdots(1+q^j)}{(1+q^N)(1+q^{N-1})\cdots(1+q^{N-j+1})} \\ &= \frac{(1-q^{2N+1})(1-q^N)(1-q^{2N-1})(1-q^{N-1})\cdots(1-q^{2N-2j+3})(1-q^{N-j+1})}{(1-q^j)(1-q^{2j-1})(1-q^{j-1})(1-q^{2j-3})\cdots(1-q)(1-q)}\end{aligned}$$

and one sees that this is the same as the expression for $\left[\begin{smallmatrix} 2N+1 \\ 2j \end{smallmatrix}\right]$ except that each even exponent in numerator and denominator has been divided by 2. Thus the divisibility properties previously described are preserved since the only change is that j numerator exponents and j denominator exponents have been divided by 2 which of course does not affect whether a denominator exponent divides a numerator exponent (i.e. if l is odd and $l|2M$ then $l|M$, if l is even and $l|2M$ then $\frac{l}{2}|M$). Thus the denominator of

$$\begin{bmatrix} 2N+1 \\ 2j \end{bmatrix} \frac{(1+q)(1+q^2)\cdots(1+q^j)}{(1+q^N)(1+q^{N-1})\cdots(1+q^{N-j+1})}$$

has no zeros that are not cancelled by those of the numerator. This proves Lemma 1. $\square$

LEMMA 2. *The q-tangent numbers satisfy*

$$T_{2N+1}(q)+\sum_{j=1}^{N}(-q)_{2j-1}\begin{bmatrix}2N+1\\2j\end{bmatrix}(-1)^jT_{2N+1-2j}(q)$$

(2.4)

$$=(-1)^N(-q)_{2N};$$

where $(a)_n=(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1}),\ (a)_0=1.$

PROOF. We have

$$\sum_{n=0}^{\infty}\frac{T_{2n+1}(q)x^{2n+1}}{(q)_{2n+1}}=\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+1}}{(q)_{2n+1}}\Big/\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n}}{(q)_{2n}}. \tag{2.5}$$

Now (here $i=\sqrt{-1}$)

$$\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+1}}{(q)_{2n+1}}=\sum_{n=0}^{\infty}\frac{i^{n-1}x^n}{(q)_n}\frac{(1-(-1)^n)}{2}$$

(2.6)

$$=\frac{1}{2i}\left(\frac{1}{(ix)_\infty}-\frac{1}{(-ix)_\infty}\right)\quad \left(\text{by }[\mathbf{1}\text{, p. 19, equation (2.2.5)}]\right);$$

$$\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n}}{(q)_{2n}}=\sum_{n=0}^{\infty}\frac{i^nx^n}{(q)_n}\frac{(1+(-1)^n)}{2}$$

(2.7)

$$=\frac{1}{2}\left(\frac{1}{(ix)_\infty}+\frac{1}{(-ix)_\infty}\right).$$

Therefore

$$\sum_{n=0}^{\infty}\frac{T_{2n+1}(q)x^{2n+1}}{(q)_{2n+1}}=\frac{1}{2i}\left(\frac{1}{(ix)_\infty}-\frac{1}{(-ix)_\infty}\right)\Big/\frac{1}{2}\left(\frac{1}{(ix)_\infty}+\frac{1}{(-ix)_\infty}\right)$$

$$=\frac{1}{i}\,\frac{(-ix)_\infty-(ix)_\infty}{(-ix)_\infty+(ix)_\infty}=\frac{1}{i}\,\frac{(-ix)_\infty/(ix)_\infty-1}{(-ix)_\infty/(ix)_\infty+1}.$$

Clearing the denominator on the right and utilizing the q-binomial series

$$\sum\frac{(A)_nz^n}{(q)_n}=\frac{(Az)_\infty}{(z)_\infty}$$

[**1**, p. 17, equation (2.2.1)], we find that

$$\left(1+\sum_{n=0}^{\infty}\frac{(-1)_n(ix)^n}{(q)_n}\right)\sum_{n=0}^{\infty}\frac{T_{2n+1}(q)x^{2n+1}}{(q)_{2n+1}}=\frac{1}{i}\sum_{n=1}^{\infty}\frac{(-1)_n(ix)^n}{(q)_n}.$$

Let us now compare the real parts of the coefficient of x^{2N+1} in this last identity:

$$2T_{2N+1}(q)+\sum_{j=1}^{N}(-1)_{2j}\begin{bmatrix}2N+1\\2j\end{bmatrix}(-1)^jT_{2N+1-2j}(q)=(-1)_{2N+1}(-1)^N,$$

and if we divide each side of this identity by 2 we obtain the result stated in Lemma 2. □

THEOREM 1. *The polynomial* $(1 + q)(1 + q^2) \cdots (1 + q^N)$ *divides the polynomial* $T_{2N+1}(q)$.

PROOF. The result is immediate for $N = 0, 1$ since $T_1 = 1$ and $T_3 = q(1 + q)$. Let us now assume the result true up to but not including N.

Now

$$\begin{aligned}(-q)_{2j-1}\begin{bmatrix}2N+1\\2j\end{bmatrix} &= (-q)_j(-q^{j+1})_{j-1}\begin{bmatrix}2N+1\\2j\end{bmatrix}\\ &= (1+q^N)(1+q^{N-1})\cdots(1+q^{N-j+1})(-q^{j+1})_{j-1}\\ &\quad\times\frac{(1+q)(1+q^2)\cdots(1+q^j)}{(1+q^N)(1+q^{N-1})\cdots(1+q^{N-j+1})}\begin{bmatrix}2N+1\\2j\end{bmatrix}.\end{aligned}$$

Hence by Lemma 1, $(1 + q^N)(1 + q^{N-1}) \cdots (1 - q^{N-j+1})$ is factor of the polynomial $(-q)_{2j-1}[\begin{smallmatrix}2N+1\\2j\end{smallmatrix}]$. By the induction hypothesis $(1 + q)(1 + q^2) \cdots (1 + q^{N-j})$ is a factor of $T_{2N+1-2j}(q)$. Hence for $1 \leqslant j \leqslant N$, we see that $(-q)_N$ is a factor of

$$(-1)^j(-q)_{2j-1}\begin{bmatrix}2N+1\\2j\end{bmatrix}T_{2N+1-2j}(q),$$

and since $(-q)_N$ is obviously a factor of $(-q)_{2N}$ we deduce from Lemma 2 that $(-q)_N$ is a factor of $T_{2N+1}(q)$ as well. Thus Theorem 1 follows by induction.

3. Conclusion. First we note that the result mentioned in the Introduction about the divisibility of the $T_{2n+1}(q)$ by the cyclotomic polynomials $\phi_2(q)$, $\phi_4(q), \ldots, \phi_{2n}(q)$ now follows from Theorem 1 since $\phi_{2n}(q)$ divides $(1 + q^n)$.

We also note that the divisibility of $T_{2n+1}(q)$ by specific factors of the form $1 + q^j$ can be handled again by Lemma 2. For example:

THEOREM 2. *The polynomial* $(1 + q)^n$ *is a factor of the q-tangent number* $T_{2n+1}(q)$.

PROOF. The result is obvious for $n = 0, 1$ since $T_1(q) = 1$ and $T_3(q) = q(1 + q)$. Assume the theorem true up to but not including n. Now since $1 + q^{2M+1} = (1 + q)(1 - q + q^2 - \cdots + q^{2M})$, we see that $(1 + q)^j$ is a factor of $(-q)_{2j-1}$. By the induction hypothesis $(1 + q)^{N-j}$ is a factor of $T_{2N+1-2j}(q)$. Hence $(1 + q)^N$ is a factor of

$$(-q)_{2j-1}\begin{bmatrix}2N+1\\2j\end{bmatrix}(-1)^jT_{2N+1-2j}(q),$$

and since $(1 + q)^N$ is also a factor of $(-q)_{2N}$, we deduce from Lemma 2 that $(1 + q)^N$ is a factor of $T_{2N+1}(q)$.

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