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NONLINEAR FREDHOLM MAPS OF INDEX ZERO AND THEIR SINGULARITIES

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ABSTRACT. Let $F: X \to Y$ be a C^1 Fredholm map of index zero between two Banach spaces. Defining the singular set $B = \{x | F'(x) \text{ is not surjective}\}$, we study the local and global effect of B on the map F. In particular it is shown that if $b \in B$ is isolated in B, then, for dim X and dim $Y > 3$, F is a local homeomorphism at b. We then show that if B consists of discrete points, F is a global homeomorphism of X onto Y. A nonlinear partial differential equation is included as an illustration.

When one formulates nonlinear elliptic partial differential equations as maps $F: X \to Y$ between Banach spaces we often encounter the class of C^1 Fredholm maps of index zero [4]. An immediate question is the existence and uniqueness of solutions, i.e., when is F a global homeomorphism. If the singular set $B = \{x | F'(x) \text{ not surjective}\}$ is empty, then the inverse function theorem guarantees that F is a local homeomorphism. In 1934 Banach and Mazur [1] showed that if in addition F is proper (Definition 1), then F is a global homeomorphism. We can ask for an extension of this result when B is nonempty.

In §I we establish some preliminary results and definitions, including a proof of the Banach-Mazur theorem for completeness. In §II we study the effects of the singular set and prove generalizations of both the inverse function theorem and the Banach-Mazur Theorem when $B \neq \varnothing$ (Theorems 4 and 5). We end with a nonlinear partial differential equation illustrating our results.

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I. Preliminaries. Let $F: X \to Y$ be a continuously differentiable map between two Banach spaces. We also suppose that F is a Fredholm map [4].

The singular set B of F is defined as: $B = \{x | F'(x) \text{ is not surjective}\}$. Its image $S = F(B)$ is called the singular values. Since the noninvertible linear maps are closed in $L(X, Y)$ (the space of linear maps of X into Y) then it follows from the continuity of the derivative map F' that B is a closed set.

Since the conditions of the inverse function theorem require that $F'(x)$ be invertible, we can reformulate this by saying that F is a C^1 Fredholm map of

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index = 0 and $B = \emptyset$. Thus we shall also restrict ourselves to the class index $F = 0$.

In order to globalize our results, we shall only consider what are called proper maps.

DEFINITION 1. F: $X \to Y$ is proper if whenever $K \subset Y$ is compact then $F^{-1}(K)$ is compact in X.

In particular if y is a point, then $F^{-1}(y)$ is certainly compact.

Also a proper map is a closed map for if $C \subset X$ is closed and the sequence $y_n \in F(C)$ converges to y, then $F^{-1}(\{y_n\} \cup y)$ is compact and so has a limit point x. By continuity $F(x) = y$.

Fundamental to our investigation is the notion of a covering space map.

DEFINITION 2. A map F of X onto Y is a covering space map if we can find a covering of Y by open sets U so that for each U, $F^{-1}(U)$ is the disjoint union of open sets each of which is mapped homeomorphically onto U.

The key to our use of covering space maps is [5]:

LEMMA 1. *If F: $X \to Y$ is a covering space map, X and Y pathwise connected and Y simply connected, then F is a global homeomorphism.*

Thus our first question is to find conditions which insure that a map is a covering space map. This is answered by [3]:

COVERING SPACE THEOREM 1. *Let $U \subset X$ be a connected open set in a Banach space X. Then F: $U \to F(U)$ is a covering space map between these two sets if* (i) *F is a local homeomorphism and* (ii) *F is proper.*

Before we discuss the effects that the singular set has on the properties of a mapping we shall need to know what happens when there is no singular set, i.e., when $B = \emptyset$. Locally the inverse function theorem tells us that the map is a local homeomorphism. The question is what happens globally. In this case the definitive result is the Banach-Mazur theorem [1]:

THEOREM 2. *In order that F: $X \to Y$ be a global homeomorphism of X onto Y it is necessary and sufficient that* (i) *F is a proper map and* (ii) *F is a local homeomorphism.*

PROOF. The necessity follows from the fact that a homeomorphism is proper since F^{-1} is continuous. The sufficiency follows, once we show that F maps X onto Y, from Theorem 1 with $U = X$ and Lemma 1 observing that a Banach space Y is certainly connected and simply connected. However the ontoness of F is assured since (i) a proper map is in particular a closed map and (ii) a local homeomorphism is an open map. Thus $F[X]$ is both open and closed and so must be all of Y by connectivity.

When F is continuously differentiable we can reformulate this theorem using the inverse function theorem.

THEOREM 2a. *In order that the C^1 map F: $X \to Y$ be a global diffeomorphism, it is necessary and sufficient that* (i) *F is proper,* (ii) *F is a*

Fredholm map of index $= 0$ *and* (iii) $B = \varnothing$.

II. The singular set. The Banach-Mazur theorem (our Theorems 2 and 2a) leads us to ask how the singular set affects the conclusions of this theorem, in particular what type of singular set insures that our map is a global homeomorphism.

THEOREM 3. *Let* $F\colon X \to Y$ *be a* C^1 *proper Fredholm map of index* $= 0$. *Then if* (i) $F(X) \neq S$ *and* (*ii*) $Y - S$ *is connected then* $F\colon X - F^{-1}(S) \to Y - S$ *is a covering space map*.

PROOF. Since F is proper and B is closed, $S = F(B)$ is closed. Thus $X - F^{-1}(S)$ and $Y - S$ are open. Since $F'(x)$ is invertible on $X - F^{-1}(S)$, we can apply the inverse function theorem to conclude that F is an open map on this set.

We next show that F is a proper map between $X - F^{-1}(S)$ and $Y - S$. This is enough because F will then be an open and closed map and so by the connectivity of $Y - S$, it will map $X - F^{-1}(S)$ onto $Y - S$. We may then apply Theorem 1 to obtain the desired conclusion.

However F is proper on $X - F^{-1}(S)$, for if $K \subset Y - S$ is compact then K is compact in Y. Thus $F^{-1}(K)$ is compact in X. However $F^{-1}(K) \subset X - F^{-1}(S)$ and so it is compact in this set.

We observe that with F and S satisfying the conditions in the above theorem, then F maps X onto Y since the theorem shows it maps $X - F^{-1}(S)$ onto $Y - S$, and trivially, $F^{-1}(S)$ onto S. Thus if S is "thin" enough, the surjectivity of F is insured and the study of F reduces to its nice behavior as a covering map on $X - F^{-1}(S)$ onto $Y - S$ and its bad behavior on $F^{-1}(S)$ onto S. In the latter case however, F is still a proper map. With further assumptions, we can study F on this bad part. In fact we show that isolated singularities are removable in the following sense:

THEOREM 4. *Let* $F\colon X \to Y$ *be a* C' *proper Fredholm map of index* $= 0$ *between two Banach spaces whose dimensions are* 3 *or greater. If* $b \in B$ *is isolated in* B *then* F *is a local homeomorphism about* b.

PROOF. Choose an open ball O about b so that $\overline{O} \cap B = b$. Let $F(b) = p$. If $x \in F^{-1}(p) \cap \overline{O}$ and $x \neq b$ then F is a local homeomorphism about x. Thus b is the only limit point of $F^{-1}(p) \cap \overline{O}$. Hence $F^{-1}(p) \cap \partial O$ is compact and so consists of only a finite number of points. Thus by enclosing these points in sufficiently small closed balls and then excising these closed balls from O, we construct an open set O' about b such that $\partial O' \cap F^{-1}(p) = \varnothing$.

About p choose open balls O_n of radius $1/n$. Let C_n be the component of $F^{-1}(O_n) \cap O'$ containing b. For some N, $\overline{C}_N \cap \partial O' = \varnothing$. If not then we can find a sequence $x_n \in C_n \cap \partial O'$. Since $F(x_n) \in \overline{O}_n$, then $F(x_n) \to p$. By the properness of F, $x_n \to \bar{x}$ and $\bar{x} \in F^{-1}(p) \cap \partial O'$. This contradicts our construction of O'. Thus $\overline{C}_N \subset O'$ for some N.

We show that F: $C_N \to O_N$ is proper. Let $K \subset O_N$ be compact. Thus K is compact in Y and so $F^{-1}(K) \cap \overline{C}_N$ is compact and furthermore it does not intersect $\partial O'$. Hence dist$(F^{-1}(K) \cap \overline{C}_N, \partial O') = \varepsilon > 0$ and so if $x \in F^{-1}(K) \cap \overline{C}_N$ there is an open ball O_x about it so that $\overline{O}_x \subset O'$ and $F(O_x) \subset O_N$ (otherwise we could find a sequence $x_n \in F^{-1}(K) \cap \overline{C}_N$, $x_n \to x$ and $F(x_n) \notin O_N$. However since $F(x_n) \to F(x)$, this would contradict the fact that $F(x)$ is an interior point of O_N). Thus $O_x \subset O' \cap F^{-1}(O_N)$. Since O_x is connected and $O_x \cap C_N \neq \emptyset$, then $O_x \subset C_N$. Hence $\overline{C}_N \cap F^{-1}(K) = C_N \cap F^{-1}(K)$ is compact. Thus F is proper on C_N into O_N.

Next we show that F: $C_N - F^{-1}(p) \to O_N - p$ is a homeomorphism. This follows from Theorem 3 and Lemma 1 after we check their hypotheses. Firstly, F is a proper map by the above argument. It is also open since there are no singular points. Hence F is onto by the connectivity of $O_N - p$. Secondly $C_N - F^{-1}(p)$ is connected. This follows in the case of infinite dimensional X **[8]** since $F^{-1}(p) \cap C_N$ is compact and in the finite dimensional and thus separable case since $F^{-1}(p) \cap C_N$ is countable and compact. Finally if dim $Y \geqslant 3$, $O_N - p$ is simply connected.

We finally show that F is 1-1 on C_N and thus a homeomorphism by the invariance of domain theorem **[6]**. We shall show that $F^{-1}(p) \cap C_N = b$.

Suppose r is a regular value of $F^{-1}(p) \cap C_N$. Then r and b are contained in disjoint open sets O_r and O_b in C_N. Furthermore we choose O_r so that F is a homeomorphism on it. Choosing a sequence $x_n \in C_N - F^{-1}(p)$ and $x_n \to b$, we can find a sequence $\overline{x}_n \to r$ and $x_n \neq \overline{x}_n$, $F(x_n) = F(\overline{x}_n)$ contradicting the univalence of F on $C_N - F^{-1}(p)$. Hence $F^{-1}(p) \cap C_N = b$.

The counterexamples $F(x) = x^2$ in $\mathbf{R}'$ and $F(z) = z^2$ in $\mathbf{R}^2$ illustrate the necessity of the restrictions on the dimensions of the spaces in Theorem 4.

In **[6]** it is shown that if F is a 1-1 proper Fredholm map with index $= 0$, then B is nowhere dense. We present a partial converse by applying Theorems 4 and 2.

THEOREM 5. *Let F: $X \to Y$ be a C' proper Fredholm map of index* 0 *between Banach spaces whose dimensions are* $\geqslant 3$. *If the singular set B is discrete, then F is a global homeomorphism of X onto Y.*

Theorem 4 essentially shows that at an isolated singularity b, locally $F^{-1}(F(b)) = b$. We globalize this relation in the case where X and Y are infinite dimensional.

THEOREM 6. *Let F: $X \to Y$ be a C' proper Fredholm map of index* 0 *between infinite dimensional Banach spaces. If the singular set B is the countable union of compact sets, then $F^{-1}(F(B)) = B$, and F is a global diffeomorphism of $X - B$ onto $Y - S$.*

PROOF. Since F is proper, $S = F(B)$ and $F^{-1}(S)$ are closed and the countable union of compact sets. By the infinite dimensionality of X and Y, $X - F^{-1}(S)$ is connected and $Y - S$ is connected and simply connected **[8]**. By Theorem 3 and Lemma 1 F is a global diffeomorphism of $X - F^{-1}(S)$

onto $Y - S$. Let $y \in S$. We look at $F^{-1}(y)$. Let $b \in F^{-1}(y) \cap B$. Suppose $r \in F^{-1}(y)$ is a regular point. Then r and b are isolated from each other and can be enclosed in disjoint open sets O_r and O_b. We choose O_r so that F is a homeomorphism on it. Since $F^{-1}(S)$ is of the first category in X, we can find a sequence $x_n \to b$ and $x_n \in X - F^{-1}(S) \cap O_b$. Since $F(x_n) \to y$, we can find points $\bar{x}_n \to r$ and $F(x_n) = F(\bar{x}_n)$, contradicting the univalence of F on $X - F^{-1}(S)$. Hence $F^{-1}(S)$ contains no regular points, i.e., $F^{-1}(S) = B$.

We remark again that Theorem 6 is purely infinite dimensional as proven. Thus the size of the space proves to be an advantage to us. I do not know if Theorem 6 is true in finite dimensions.

As an illustration of our results, let us look at the equation:

$$(*) \qquad \begin{aligned} &\Delta u + \lambda_1 u - u^3 = f\,(f \in L^2). \\ &u = 0 \text{ on the boundary of } \Omega. \end{aligned}$$

Here Ω is a bounded domain in $\mathbf{R}^N$ and λ_1 is the smallest positive eigenvalue of Δ on Ω. We reformulate $(*)$ as a map $F(u)$ on the Sobolev space $\mathring{W}_{1,2}$ where $(F(u), v)_{1,2} = \int \nabla u \cdot \nabla v - \lambda_1 \int uv + \int u^3 v$. In order to avoid excessive detail we assume $N \leqslant 3$ is chosen so that results involving the Sobolev imbedding theorems and regularity of solutions are justified. We refer the reader to **[2]** for a treatment of techniques and theorems used in the functional analytic approach to partial differential equations.

On $\mathring{W}_{1,2}$ $F(u)$ has a derivative $F'(u)v = \Delta v + \lambda_1 v - u^2 v$. Thus $F'(u)$ is a selfadjoint elliptic operator and so $F(u)$ is a Fredholm map of index zero. Also if $F'(u)v = 0$, then using the variational characterization of λ_1, we have $0 = (F'(u)v, v) = \int |\nabla v|^2 - \lambda_1 v^2 + \int u^2 v^2 > 0$ unless $u = 0$. In this case $F'(0)v = 0$ is the problem $\Delta v + \lambda_1 v = 0$, $v = 0$ on the boundary of Ω. Since this problem has nontrivial solutions we conclude that for $F(u)$, the singular set $B = \{0\}$.

We now show F is proper. First the inverse image of a bounded set is bounded. To see this we prove the equivalent statement: $\|F(u)\|_{1,2} \to \infty$ as $\|u\|_{1,2} \to \infty$. So suppose $\|u\|_{1,2}^2 = \|\nabla u\|_2^2 + \|u\|_2^2 \to \infty$. If $\|\nabla u\|_2^2 \to \infty$ and $\|u\|_2^2 \leqslant k$, then using the fact that $\|\nabla u\|_2^2 \geqslant c\|u\|_2^2$ we hae $(F(u), u) = \int |\nabla u|^2 - \lambda_1 u^2 + \int u^4 \geqslant \|\nabla u\|_2^2 - \lambda_1 k$. Thus for $\|u\|_{1,2}^2$ large enough, $(F(u), u)/\|u\|_{1,2}^2 \geqslant \gamma > 0$. If $\|u\|_2^2 \to \infty$ also, then $(F(u), u) \geqslant \|\nabla u\|_2^2 + (\beta\|u\|_2^2 - \lambda_1)\|u\|_2^2$ (since $L^4 \subset L^2$). Thus if $\|u\|_{1,2} \to \infty$, then $(F(u), u)/\|u\|_{1,2}^2 \geqslant \varepsilon > 0$.

Applying the Cauchy-Schwarz inequality we can conclude that $\|F(u)\|_{1,2} \to \infty$ when $\|u\|_{1,2} \to \infty$.

So to show F is proper, let $K \subset \mathring{W}_{1,2}$ be compact. By the above result $F^{-1}(K)$ is bounded. Thus if u_n is a sequence in $F^{-1}(K)$, then $\{u_n\}$ has a weakly convergent subsequence since $\mathring{W}_{1,2}$ is a Hilbert space. As $F(u_n)$ lies in the compact set K, it has a convergent subsequence. Renumbering, we have $u_n \to u$ weakly and $F(u_n) \to y$ strongly. However if $N \leqslant 3$, we can apply the Sobolev theorems governing compact imbeddings to conclude that $u_n \to u$ strongly in L^2 and L^4. Thus since $(F(u_n), u_n)$ converges and since $(F(u_n),$

$u_n) = \|\nabla u_n\|_2^2 - \lambda_1 \|u_n\|_2^2 + \|u_n\|_4^4$, then we conclude that $\|\nabla u_n\|_2$ converges. Thus we are in the situation where u_n converges weakly and $\|u_n\|_{1,2}$ converges in a Hilbert space. Thus $u_n \to u$ strongly, i.e., every sequence in $F^{-1}(K)$ has a convergent subsequence. Thus F is proper.

We now apply Theorem 5 to conclude that F is a global homeomorphism of $\mathring{W}_{1,2}$ onto itself, and so our problem $(*)$ has, for every $f \in L^2$, a unique solution depending continuously on f.

A natural question to ask is for generalizations to Fredholm maps with index $p > 0$. However in this case the results are radically different. For example it can be shown that there is no equivalent statement of the Banach-Mazur Theorem 2a. In fact if F is a proper C^1 Fredholm map with positive index, then B cannot be empty, i.e., such maps always have singularities. This result and a general investigation of the positive index case will appear in [7].

References

1. S. Banach and S. Mazur, *Über mehrdeutige stetige Abbildungen*, Studia Math. **5** (1934), 174–178.

2. M. S. Berger, *Nonlinearity and functional analysis*, Academic Press, New York, 1977.

3. R. Plastock, *Homeomorphisms between Banach spaces*, Trans. Amer. Math. Soc. **200** (1974), 169–183.

4. S. Smale, *An infinite dimensional version of Sard's theorem*, Amer. J. Math. **87** (1965), 861–866.

5. E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

6. A. J. Tromba, *Some theorems on Fredholm maps*, Proc. Amer. Math. Soc. **34** (1972), 578–585.

7. M. S. Berger and R. Plastock, *On the singularities of nonlinear Fredholm operators*, Bull. Amer. Math. Soc. **83** (1977), 1316–1318.

8. W. H. Cutler, *Negligible subsets of infinite dimensional Fréchet manifolds*, Proc. Amer. Math. Soc. **23** (1969), 668–675.

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