PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 68, Number 3, March 1978

A SHORT PROOF OF THE DAWKINS-HALPERIN THEOREM

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ABSTRACT. A brief proof is presented, of the Dawkins-Halperin Theorem, that if D is a finite dimensional division algebra with centre F, then the direct limits of appropriately-sized matrix rings over D and F are isomorphic; the isomorphism can be given in a form suitable for comparing cohomology groups of D and F.

For a ring R, we denote the ring of t by t matrices with entries from R, by M_tR. There is an obvious map from R to M_tR:

$$\Delta(t)\colon R \to M_tR, \quad r \mapsto \begin{bmatrix} r & & & & \\ & r & & & \\ & & \ddots & & \\ & & & r & \\ & & & & r \end{bmatrix}.$$

Let D denote a division ring of dimension n^2 over its centre F. The main result of [1], asserts that if $t = n^2$, then as F-algebras

$$\lim_{i\to\infty} M_{t^i}D \simeq \lim_{i\to\infty} M_{t^i}F,$$

the maps in both limits being $\Delta(t)$. However, the proof there is exceptionally obscure and complicated. We give a short natural proof, requiring only the Noether-Skolem Theorem:

[3, THEOREM 4.3.1]. *Let C be a finite dimensional simple F-algebra with centre F, and let A, B be simple subalgebras of C, each with centre F. Then any F-algebra isomorphism from A to B can be extended to an inner automorphism of C.*

THEOREM. *Let D be a division algebra of dimension n^2 over its centre, the field F. Set $t = n^2$, and for each positive integer i, let j_i denote the map $M_{t^i}F \subset M_{t^i}D$ induced by the inclusion of F in D. Form the F-algebras $\lim_{i\to\infty} M_{t^i}F$, $\lim_{i\to\infty} M_{t^i}D$, with $\Delta(t)$ as the maps in the limits.*

There exist inner automorphisms ψ_i, φ_i of $M_{t^i}D$, $M_{t^i}F$ respectively, so that if $\alpha_i\colon M_{t^i}F \to M_{t^i}D$ are defined by $\alpha_i = \psi_i^{-1} j_i \varphi_i$, then the α_i are compatible with the maps in the direct limits, and the induced map

$$\lim_{i\to\infty} \alpha_i\colon \lim M_{t^i}F \to \lim M_{t^i}D$$

is an F-algebra isomorphism.

Received by the editors May 31, 1977.
AMS (MOS) subject classifications (1970). Primary 16A40.

Let $k\colon D \to M_tF$ be a fixed F-algebra homomorphism (*for instance, the right regular representation of D*), *and define maps $k_i\colon M_{t^i}D \to M_{t^{i+1}}F$ to be the maps on the matrix rings induced by k. We may form the limit, S, of the diagram* (1):

$$F \xrightarrow{j_0} D \xrightarrow{k} M_tF \xrightarrow{j_1} M_tD \xrightarrow{k_1} M_{t^2}F \xrightarrow{j_2} \ldots . \tag{1}$$

Then S is algebra isomorphic to $\lim_{i\to\infty} M_{t^i}F$.

PROOF. Pick a fixed map k, such as the right regular representation, and separate (1) into two rows, rows 2 and 3 of diagram (2).

$$\begin{tikzcd}
\cdots \arrow[r] & M_{t^i}F \arrow[r] & M_{t^{i+1}}F \arrow[r] & M_{t^{i+2}}F \arrow[r] & \cdots \\
\cdots \arrow[r] & M_{t^i}F \arrow[r, "k_ij_i"] \arrow[d, "j_i"'] & M_{t^{i+1}}F \arrow[r, "k_{i+1}j_{i+1}"] \arrow[d, "j_{i+1}"'] & M_{t^{i+2}}F \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & M_{t^i}D \arrow[r, "j_{i+1}k_i"] & M_{t^{i+1}}D \arrow[r] & M_{t^{i+2}}D \arrow[r] & \cdots \\
\cdots \arrow[r] & M_{t^i}D \arrow[r] & M_{t^{i+1}}D \arrow[r] & M_{t^{i+2}}D \arrow[r] & \cdots
\end{tikzcd} \tag{2}$$

Because $M_{t^i}F$ and $M_{t^i}D$ are cofinal in diagram (1), $\lim j_i$ is actually an isomorphism (with inverse, $\lim k_i$) from the limit of row 2 to the limit of row 3. We shall construct inner $\varphi_i\colon M_{t^i}F \to M_{t^i}F$ (row 1 to row 2) and $\psi_i\colon M_{t^i}D \to M_{t^i}D$ (row 4 to row 3) so that the whole of (2) commutes.

Define $\psi_0\colon D \to D$ to be the identity map. Assuming ψ_s have been defined for $0 \leqslant s \leqslant i$, so that rows 3 and 4 commute, we see

$$j_{i+1}k_i\psi_i(M_{t^i}D) \simeq \Delta(M_{t^i}D)$$

as F-subalgebras of $M_{t^{i+1}}D$, the isomorphism obtained by pulling back the image of Δ, and applying $j_{i+1}k_i\psi_i$. By the Noether-Skolem Theorem, there exists an invertible V in $M_{t^{i+1}}D$ so that this isomorphism is implemented by conjugation with V. Define $\psi_{i+1}(A) = VAV^{-1}$; then $\psi_{i+1}\Delta = j_{i+1}k_i\psi_i$, concluding the induction.

Thus $\lim \psi_i$ defines a map between the limits of the fourth and third rows; it follows that $\lim \psi_i^{-1}$ (from row 3 to row 4) exists and is the inverse. In particular, $\lim \psi_i^{-1}$ is an algebra isomorphism.

The same process allows us to construct a similar isomorphism, $\lim \varphi_i$ from the limit of row 1 to the limit of row 2, with each φ_i inner. Since $\lim j_i$ is an isomorphism, and

$$\lim\left(\psi_i^{-1}j_i\varphi_i\right) = \left(\lim \psi_i^{-1}\right)(\lim j_i)(\lim \varphi_i),$$

setting $\alpha_i = \psi_i^{-1}j_i\varphi_i$ (mapping down the columns of (2)), we see that $\lim \alpha_i$ is an isomorphism, and the final statement is an immediate consequence. $\square$

The form of the isomorphism obtained above is particularly useful in computing the homology or cohomology of D relative to that of F (see [**2**], for an application), because such functors usually commute with direct limits, and change inner automorphisms into the identity.

Theorem 2 of [1] effectively asserts that if a division ring D can be represented as a limit, $\lim D_i$, with each D_i a finite dimensional central division algebra over F, then

$$\lim_{\Delta(m)} M_n F \simeq \lim_{\Delta(m)} M_n D,$$

where m varies over all products of numbers of the form $[D_i\colon F]$, and n varies similarly. (Of course, $\Delta(m)\colon M_n F \to M_r F$ is defined only if $r = mn$.) One can easily prove that this follows from our theorem above.

References

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