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SOME "ALMOST-DOWKER" SPACES

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Abstract. Call X an AD-space (for "almost-Dowker") if it is T_3 but not countably metacompact. We construct, without set-theoretic assumptions, a class of zero-dimensional, orthocompact, nonnormal AD-spaces. Using the same techniques, we simplify an example due to Hayashi by showing that if $\exp(\exp(\omega)) = \exp(\omega_1)$, (e.g., if the continuum hypothesis holds), the "Cantor tree of height ω_1" is also such a space.

Since $X \times [0, 1]$ is orthocompact iff X is orthocompact and countably metacompact, we now have "absolute" examples of orthocompact Tikhonov spaces whose products with [0, 1] are not orthocompact.

0. **Introduction**. Although completely settled in [7], it was for many years an open question whether every T_4-space is countably paracompact. In [6], M. E. Rudin had shown that if there is a Suslin tree, then there is a Dowker space, i.e., a counterexample to the above conjecture, thereby bringing the question into the realm of set theory. Then, in [4], Jensen announced that there are Suslin trees in the constructible universe, from which it follows that the existence of Dowker spaces is at least consistent with ZFC (Zermelo-Fraenkel set theory with the axiom of choice). However, it was later shown [10] that the nonexistence of Suslin trees is also consistent with ZFC. Fortunately, at about the same time the example of [7], which "exists" in ZFC alone, was discovered. Since then one or two other Dowker spaces have been constructed–for example, a first countable one of power ω_1 whose construction requires only the continuum hypothesis by way of extra set-theoretic hypothesis [8]–but all of them are quite complicated, and all but that of [7] are "consistency examples", i.e., they require additional set-theoretic assumptions beyond the axioms of ZFC.

In view of the apparent scarcity of Dowker spaces, one might naturally look also for "almost-Dowker" spaces; the problem is to decide just what such a space should be. It is not hard to find examples of Tikhonov spaces which are not countably paracompact: the familiar Cantor tree and the hyperspace of the integers, for example. However, countable paracompactness is, in the presence of normality, equivalent to countable metacompactness, so we shall set our sights a little higher.

0.0. Definition. An *AD-space* is a T_3-space which is not countably metacompact.

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It turns out that AD-spaces are almost as hard to come by as Dowker spaces. The Cantor tree, for example, is a Moore space, hence perfect (closed sets are G_δ-sets), hence countably metacompact; and the countable metacompactness of 2^N remains undetermined. Example 4.2 of [9] is Hausdorff and not countably metacompact, but it fails utterly to be regular. In [3], however, Hayashi constructs–on the assumption, consistent with ZFC, that $2^{2\lambda} = 2^\lambda$ for some regular $\lambda > \omega$–a (nonnormal) AD-space.

The purpose of this paper is, by combining Hayashi's technique with a result of Fleissner in [2], to produce an "absolute" example of a (nonnormal) AD-space. As a by-product, we shall also obtain a "consistency example" closely related to (but simpler than) Hayashi's.

1. **Notation, conventions and preliminaries.** All spaces in this paper are assumed to be at least T_1. Ordinals are von Neumann ordinals, (i.e., an ordinal *is* the set of smaller ordinals,) and are denoted by lower-case Greek letters. Cardinals are initial ordinals and are denoted by κ and λ. If X is a space, $\tau(X)$ denotes the topology of X. If κ is a cardinal, a G_κ-set in a space X is one which is the intersection of κ members of $\tau(X)$; G_ω-sets are, as usual, referred to as G_δ-sets. X is *κ-open* if every G_λ-set in X, for $\lambda < \kappa$, is open in X; every space, therefore, is ω-open. If A and B are sets, ${}^AB$ is the set of functions from A into B. Functions are viewed as sets of ordered pairs, and, if f is a function, $\operatorname{dom} f$ and $\operatorname{ran} f$ are, respectively, its domain and range; thus $f \subseteq g$ iff $\operatorname{dom} f \subseteq \operatorname{dom} g$ and $g \upharpoonright \operatorname{dom} f = f$. A space X is *κ-Baire* iff every intersection of κ dense, open subsets of X is dense in X, (so that "ω-Baire" is just the usual notion of "Baire").

1.0. Definition. Let κ and λ be cardinals, with $\lambda \geqslant \omega$ and regular. Then $Z(\kappa, \lambda)$ is the space whose underlying set is ${}^\lambda\kappa$, and whose topology is generated by the base $\mathfrak{B}(\kappa, \lambda) = \{B(f): f \in \bigcup\{{}^\alpha\kappa: \alpha \in \lambda\}\}$, where $B(f) = \{x \in Z(\kappa, \lambda): f \subseteq x\}$. It is easily seen that $Z(\kappa, \lambda)$ is always λ-Baire. (Indeed, it follows from [5] that $Z(\kappa, \lambda)$ is non-Archimedean, λ-metrizable, and hereditarily ultraparacompact, for example.)

1.1. Definition. Let κ and λ be as above, and let $X \subseteq Z(\kappa, \lambda)$. Then $\mathfrak{B}_X(\kappa, \lambda) = \{B \in \mathfrak{B}(\kappa, \lambda): B \cap X \neq \varnothing\}$, and $T(X)$ is the space whose underlying set is $\mathfrak{B}_X(\kappa, \lambda) \cup X$, topologized as follows: each element of $\mathfrak{B}_X(\kappa, \lambda)$ is an isolated singleton, and a base at a point $x \in X$ is given by $\mathfrak{N}(x) = \{N(x, \alpha): \alpha \in \lambda\}$, where $N(x, \alpha) = \{x\} \cup \{B(f) \in \mathfrak{B}_X(\kappa, \lambda): f \subseteq x \;\&\; \alpha \subseteq \operatorname{dom} f\}$. Clearly, $T(X)$ is λ-open, Tikhonov, zero-dimensional, and–for later reference–orthocompact (i.e., every open cover, $\mathcal{V}$, of $T(X)$ has an open refinement, $\mathfrak{R}$, which covers $T(X)$ and has the property that if $\mathfrak{U} \subseteq \mathfrak{R}$, then $\bigcap \mathfrak{U} \in \tau(T(X))$).

The following proposition is hardly more than a restatement of the definition of λ-Baireness.

1.2. Lemma. *Let κ and λ be cardinals, with $\lambda \geqslant \omega$ and regular, and let X be a λ-Baire subspace of $Z(\kappa, \lambda)$. Let Y be a λ-Baire, dense subspace of X, and* *suppose that* $Y = \bigcup\{Y_\alpha: \alpha \in \lambda\}$. *Then, for any* $B(f) \in \mathfrak{B}_X(\kappa, \lambda)$, *there are a* $B(g) \in \mathfrak{B}_X(\kappa, \lambda)$ *and an* $\alpha \in \lambda$ *such that* $B(g) \subseteq B(f)$ (*i.e.*, $f \subseteq g$,) *and* Y_α *is dense in* $B(g) \cap Y$.

(Lemma 1.2 is obviously a special case of a more general result; it is stated in this form for future use.)

The key result is the following.

1.3. THEOREM. *Let* κ, λ *and* X *be as in Lemma* 1.2, *but with* $\lambda > \omega$, *and suppose that* $X = \bigcup\{X_n: n \in \omega\}$ *where each* X_n *is* λ-*Baire and dense in* X, *and* $X_n \cap X_m = \varnothing$ *whenever* $n < m < \omega$; *then* $T(X)$ *is not countably metacompact* (*and is therefore an AD-space*).

PROOF. For $n \in \omega$, let $F_n = \bigcup\{X_i: n \leqslant i < \omega\}$, a closed subset of $T(X)$; then the F_n's are decreasing and have empty intersection, so it suffices to show that if, for each $n \in \omega$, $F_n \subseteq W_n \in \tau(T(X))$, then $\bigcap\{W_n: n \in \omega\} \neq \varnothing$. Suppose, therefore, that the W_n's are as stated; then there is a function ϕ: $X \to \lambda$ such that, for each $x \in X$, $N(x, \phi(x)) \subseteq W_n$ whenever $x \in F_n$. For each $n \in \omega$ and $\alpha \in \lambda$, let $X_n(\alpha) = \{x \in X_n: \phi(x) < \alpha\}$.

Repeated applications of Lemma 1.2 now allow us inductively to construct sets $\{B(f_n): n \in \omega\} \subseteq \mathfrak{B}_X(\kappa, \lambda)$ and $\{\alpha_n: n \in \omega\} \subseteq \lambda$ such that for every $n \in \omega$, $X_n(\alpha_n)$ is dense in $B(f_n) \cap X$, $\alpha_n \subseteq \operatorname{dom} f_n$, $\alpha_{n+1} > \alpha_n$, and $B(f_{n+1}) \subseteq B(f_n)$. Let $\alpha = \sup\{\alpha_n: n \in \omega\}$, let $f = \bigcup\{f_n: n \in \omega\}$, and choose $B(g) \in \mathfrak{B}_X(\kappa, \lambda)$ so that $B(g) \subseteq B(f)$ and $\alpha \in \operatorname{dom} g$.

Then $X_n(\alpha_n)$ is dense in $B(g) \cap X$ for each $n \in \omega$, so, in particular, there is an $x_n \in X_n(\alpha_n) \cap B(g)$. Thus, for each $n \in \omega$, $\phi(x_n) < \alpha_n < \alpha \in \operatorname{dom} g$, and $g \subseteq x_n$, so–in $T(X)$–$B(g) \in N(x_n, \phi(x_n)) \subseteq W_n$. Thus, $\bigcap\{W_n: n \in \omega\} \neq \varnothing$, and $T(X)$ is an AD-space.

An even simpler application of the same idea yields the next theorem.

1.4. THEOREM. *Let* κ, λ, *and* X *be as in Lemma* 1.2, *and suppose that* $X = X_0 \cup X_1$, *where* X_0 *and* X_1 *are disjoint, and each is* λ-*Baire and dense in* X; *then* X_0 *and* X_1 *are disjoint closed subsets of* $T(X)$ *which cannot be separated by disjoint open sets, and* $T(X)$ *is not normal.*

Having set the stage, we proceed to the examples.

2. **A "consistency example".** For the printer's sake we shall write $\exp(\lambda)$ and $\operatorname{wexp}(\lambda)$, respectively, to denote 2^λ and $2^{\underset{\sim}{\lambda}}$ ($= \sup\{\exp(\kappa): \lambda < \lambda\}$).

2.0. THEOREM. *If* $\lambda > \omega$ *is regular, and* $\exp(\operatorname{wexp}(\lambda)) = \exp(\lambda)$, *then* $T(Z(2, \lambda))$ *is a nonnormal AD-space.*

PROOF. The proof is a simple application of Theorems 1.3 and 1.4. We first observe that $|\mathfrak{B}(2, \lambda)| = \lambda \cdot \operatorname{wexp}(\lambda) = \operatorname{wexp}(\lambda)$, so that $|\tau(Z(2, \lambda))| \leqslant \exp(\operatorname{wexp}(\lambda)) = \exp(\lambda)$. Let $\mathcal{G} = \{G \subseteq Z(2, \lambda): G$ is a dense G_λ-set$\}$; then $|\mathcal{G}| \leqslant |\tau(Z(2, \lambda))|^\lambda \leqslant (\exp(\lambda))^\lambda = \exp(\lambda)$. Thus $\mathfrak{B}(2, \lambda) \times \mathcal{G} \times \omega$ can be enumerated in type $\exp(\lambda)$ as $\{\langle B(f_\alpha), G_\alpha, n_\alpha\rangle: \alpha < \exp(\lambda)\}$.

Now, for $\alpha < \exp(\lambda)$, inductively choose $x_\alpha \in (B(f_\alpha) \cap G_\alpha) \setminus \{x_\beta: \beta <$

$\alpha\}$. That this is always possible follows from the

LEMMA. *If* $B(f) \in \mathfrak{B}(2, \lambda)$ *and* $G \in \mathcal{G}$, *then* $|B(f) \cap G| = \exp(\lambda)$.

Assuming the lemma, put $X_n = \{x_\alpha: n_\alpha = n\}$ for each $n \in \omega$. Clearly, each X_n is a dense, λ-Baire subspace of $Z(2, \lambda)$, and we now apply Theorems 1.3 and 1.4.

To prove the lemma, let $\mathcal{V} = \{V_\alpha: \alpha \in \lambda\} \subseteq \tau(Z(2, \lambda))$ be such that $G = \bigcap \mathcal{V}$, and let S be the set of sequences of 0's and 1's of length less than λ. We shall inductively choose, for $s \in S$, basic sets $B(f_s) \in \mathfrak{B}(2, \lambda)$ such that

(i) $f_{\langle\,\rangle} = f$, where $\langle\,\rangle$ is the empty sequence;

(ii) $f_s \subseteq f_t$ whenever $s, t \in S$ with s an initial segment of t;

(iii) $B(f_{s\frown 0}) \cap B(f_{s\frown 1}) = \varnothing$ for each $s \in S$, where $s \frown i$ is the sequence formed by concatenating s with $\langle i \rangle$; and

(iv) if $s \in S$ is of length α, then $B(f_s) \subseteq \bigcap\{V_\beta: \beta < \alpha\}$.

This is easily done; given f_s, find a g_s such that $B(g_s) \subseteq B(f_s) \cap V_\alpha$, where α is the length of s, and then choose $B(f_{s\frown 0})$ and $B(f_{s\frown 1})$ so that they are disjoint and contained in $B(f_s)$; and if $s \in S$ is of length α for some limit ordinal α, and f_t has been chosen for each initial segment, t, of s, let $B(f_s) = \bigcap\{B(f_t): t \text{ is an initial segment of } s\} \in \mathfrak{B}(2, \lambda)$. Clearly, (i)–(iv) are satisfied.

Now let $X = \{x \in Z(2, \lambda): \forall\alpha \in \lambda \exists s \in S\ (f_s \subseteq x \,\&\, \alpha \subseteq \operatorname{dom} f_s)\}$; (ii) and (iii) guarantee that $|X| = \exp(\lambda)$, and (i), (ii), and (iv) guarantee that $X \subseteq B(f) \cap G$.

2.1. COROLLARY. *If the continuum hypothesis holds, i.e., if* $\exp(\omega) = \omega_1$, *then* $Z(2, \omega_1)$ *is a nonnormal AD-space.*

PROOF. If $\exp(\omega) = \omega_1$, then, since $\operatorname{wexp}(\omega_1) = \exp(\omega)$, $\exp(\operatorname{wexp}(\omega_1)) = \exp(\exp(\omega)) = \exp(\omega_1)$.

However, it is also consistent with ZFC that, say, $\exp(\omega) = \omega_3$ and $\exp(\omega_1) = \exp(\omega_3) = \omega_5$, so the continuum hypothesis is by no means necessary for the above result. Similarly, we have:

2.2. COROLLARY. *If the generalized continuum hypothesis holds, then* $Z(2, \lambda)$ *is a nonnormal AD-space whenever* $\lambda > \omega$ *is regular.*

3. **An "absolute" example.** We first generalize some familiar notions pertaining to ordinals.

3.0. DEFINITION. Let κ and λ be regular cardinals, with $\omega \leqslant \lambda < \kappa$; a set $S \subseteq \kappa$ is λ-*cub* iff $|S| = \kappa$ and, whenever $\langle \alpha_\xi: \xi \in \lambda \rangle$ is a strictly increasing λ-sequence of members of S, $\sup\{\alpha_\xi: \xi \in \lambda\} \in S$; S is λ-*stationary* iff S meets every λ-cub subset of κ.

3.1. PROPOSITION. *Let* κ *and* λ *be as in Definition* 3.0, *and let* $\mathcal{C}$ *be a family of fewer than* κ λ*-cub subsets of* κ; *then* $C_0 = \bigcap \mathcal{C}$ *is* λ*-cub.*

The proof of Proposition 3.1 is a straightforward modification of the proof that the intersection of fewer than κ cub subsets of a regular $\kappa > \omega$ is cub.

Proposition 3.1, together with the method of [11], yields the following theorem.

3.2. THEOREM. *Let κ and λ be cardinals, with $\omega \leqslant \lambda \leqslant \kappa$ and λ regular, and let $S \subseteq \kappa^+$ (the next cardinal after κ) be λ-stationary. Then S can be partitioned into $\kappa^+ \lambda$-stationary sets.*

3.3. DEFINITION. Let α and β be ordinals, with $\alpha \leqslant \beta$, and let $f \in {}^{\alpha}\beta$; f is *normal* iff $f(\xi) < f(\eta)$ whenever $\xi < \eta < \alpha$, and $f(\eta) = \sup\{f(\xi)\colon \xi < \eta\}$ whenever $\eta < \alpha$ is a limit ordinal. If $\lambda > \omega$ is regular, we let $X_\lambda = \{x \in Z(\mathrm{w}\exp(\lambda)^+, \lambda)\colon x \text{ is normal}\}$.

The following generalizes a result of Fleissner's [2].

3.4. THEOREM. *Let $\lambda > \omega$ be regular, let $\kappa = \mathrm{wexp}(\lambda)^+$, let $S \subseteq \kappa$ be λ-stationary, and let $X_\lambda(S) = \{x \in X_\lambda\colon \hat{x} \in S\}$, where $\hat{x} = \sup \operatorname{ran} x$. Then $X_\lambda(S)$ is λ-Baire and dense in X_λ.*

PROOF. Only the first assertion requires proof. Let $\mathcal{F} = \{f\colon B(f) \in \mathcal{B}(\kappa, \lambda) \,\&\, f \text{ is normal}\} = \{f\colon B(f) \in \mathcal{B}_{X_\lambda}(\kappa, \lambda)\}$. Let $\mathcal{V} = \{V_\alpha\colon \alpha \in \lambda\} \subseteq \tau(X_\lambda)$ be a family of dense sets, and fix $f \in \mathcal{F}$ arbitrarily; we must show that $X_\lambda(S) \cap B(f) \cap \bigcap \mathcal{V} \neq \varnothing$.

For each $\alpha \in \lambda$, let $\Sigma_\alpha = \{\sigma \in \mathcal{F}^\alpha\colon f \subseteq \sigma(0) \,\&\, \sigma(\xi) \subseteq \sigma(\eta)$ whenever $\xi < \eta < \alpha \,\&\, \forall \xi \in \alpha(B(\sigma(\xi)) \subseteq V_\xi)\}$, and let $\Sigma = \bigcup\{\Sigma_\alpha\colon \alpha \in \lambda\}$. For each $\sigma \in \Sigma$, let $f_\sigma = \bigcup\{\sigma(\xi)\colon \xi \in \operatorname{dom} \sigma\}$, and let $\hat{\sigma} = \hat{f}_\sigma$. Since the members of $\mathcal{V}$ are dense and open in X_λ, there is a function $\phi\colon \Sigma \times \kappa \to \Sigma$ with the following property: if $\sigma \in \Sigma$, $\eta \in \kappa$, and $\tau = \phi(\sigma, \eta)$, then $\sigma \subseteq \tau$ and $\hat{\tau} > \eta$.

For $\alpha \in \kappa$, let $\Pi_\alpha = \{\langle \sigma, \eta \rangle \in \Sigma \times \alpha\colon \hat{\sigma} < \alpha\}$, and note that $|\Pi_\alpha| \leqslant \mathrm{wexp}(\lambda) < \kappa$. Thus, there is a function $\psi\colon \kappa \to \kappa$ such that, for each $\alpha \in \kappa$, if $\langle \sigma, \eta \rangle \in \Pi_\alpha$ and $\tau = \phi(\sigma, \eta)$, then $\hat{\tau} < \psi(\alpha)$. Let $C = \{\alpha \in \kappa\colon \forall \beta < \alpha\ (\psi(\beta) < \alpha)\}$; it is easily seen that C is cub in κ. Thus, there is an $\eta \in S \cap C$ such that $cf(\eta) = \lambda$.

Let $\langle \eta_\xi\colon \xi \in \lambda \rangle$ converge up to η. Fix $\sigma_0 \in \Sigma$ arbitrarily with $\hat{\sigma}_0 < \eta$. Given $\sigma_\xi \in \Sigma$ (for some $\xi \in \lambda$) with $\hat{\sigma}_\xi < \eta$, let $\sigma_{\xi+1} = \phi(\sigma_\xi, \eta_\xi)$; and if $\xi \in \lambda$ is a limit ordinal and $\sigma_\nu \in \Sigma$ with $\hat{\sigma}_\nu < \eta$ has been defined for each $\nu < \xi$ so that $\sigma_\nu \subseteq \sigma_\mu$ whenever $\nu < \mu < \xi$, let $\sigma_\xi = \bigcup\{\sigma_\nu\colon \nu < \xi\}$.

Finally, let $\sigma = \bigcup\{\sigma_\xi\colon \xi \in \lambda\}$, and let $x = \bigcup\{\sigma(\xi)\colon \xi \in \lambda\}$; obviously $x \in X_\lambda(S) \cap B(f) \cap \bigcap \mathcal{V}$.

It is now a simple matter to combine Theorems 1.3, 1.4, 3.2 and 3.4 to get:

3.5. THEOREM. *Let λ and X_λ be as in Theorem 3.4; then $T(X_\lambda)$ is a nonnormal AD-space.*

4. **Remarks**. Dowker showed in [1] that the countably paracompact ones are precisely the normal spaces whose products with the interval [0, 1] are normal. Similarly, it was shown in [9] that the countably metacompact ones are precisely the orthocompact spaces whose products with [0, 1] are orthocompact. Since the orthocompactness of none of the known Dowker spaces

has been determined, Theorem 3.5 gives the first "absolute" example of a regular (in fact, zero-dimensional), orthocompact space whose product with [0, 1] is not orthocompact.

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