PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 68, Number 3, March 1978

A GENERALIZATION OF VON NEUMANN'S INEQUALITY TO THE COMPLEX BALL[1]

S. W. DRURY

ABSTRACT. A necessary and sufficient condition is found for a polynomial Q of J variables to be such that $Q(A_1, \ldots, A_J)$ is a contraction whenever A_j $(1 \leqslant j \leqslant J)$ are commuting linear operators on complex hilbert space satisfying $\sum_{j=1}^{J} A_j^* A_j \leqslant I$.

1. Introduction and notations. A celebrated theorem of J. von Neumann [1] asserts that, for A a linear contraction on a complex hilbert space and Q a complex polynomial, the inequality

$$\|Q(A)\| \leqslant \sup\{|Q(z)|;\, z \in \mathbf{C},\, |z| \leqslant 1\} \tag{1}$$

holds. In this note we give a generalization of this inequality to an analogous situation in several variables in which the disc is replaced by the complex ball.

Let J be a fixed integer with $J \geqslant 1$. We shall consider J-tuples $A = (A_1, \ldots, A_J)$ of linear operators on a complex hilbert space H satisfying

$$[A_{j_1}, A_{j_2}] = 0 \quad (1 \leqslant j_1, j_2 \leqslant J) \quad \text{and} \quad \sum_{j=1}^{J} \|A_j h\|^2 \leqslant \|h\|^2 \ \forall h \in H. \tag{2}$$

For $n = (n_1, \ldots, n_J)$ a multi-index (that is an element of $\mathbf{Z}_+^J$), we shall denote $|n| = \sum_{j=1}^{J} n_j$, $n! = \prod_{j=1}^{J}(n_j!)$, $\beta(n) = |n|!(n!)^{-1}$ the 'multinomial coefficients', and $\alpha(n) = (J + |n| - 1)!(n!(J-1)!)^{-1}$. If also $z = (z_1, \ldots, z_J)$ and $w = (w_1, \ldots, w_J)$ are elements of $\mathbf{C}^J$, we shall use the notations

$$|z|^2 = \sum_{j=1}^{J} |z_j|^2, \quad z \cdot w = \sum_{j=1}^{J} z_j w_j, \quad z^n = \prod_{j=1}^{J} z_j^{n_j}.$$

Similar notations will be used for a J-tuple A satisfying (2). Note that A^n makes sense since the A_j commute, and for Q a complex polynomial of J variables we will denote $Q(A) = Q(A_1, \ldots, A_J)$. Our main result is that there is an 'archetypal' J-tuple S satisfying (2) with the universal property described as follows.

THEOREM. *Let A be a J-tuple satisfying* (2) *and let Q be a complex*

Received by the editors November 17, 1976 and, in revised form, May 13, 1977.
AMS (MOS) subject classifications (1970). Primary 47A60.
Key words and phrases. Functional calculus, operators on hilbert space.
[1]The research for this article was sponsored by the National Research Council of Canada.

polynomial of J-variables; then

$$\|Q(A)\| \leqslant \|Q(S)\|. \tag{3}$$

We will give an alternative description of $\|Q(S)\|$ in §3 which makes it clear that (3) reduces to (1) when $J = 1$. Now we describe the J-tuple S explicitly. Let $l^2(\mathbf{Z}_+^J, \beta)$ stand for the complex hilbert space of functions f: $\mathbf{Z}_+^J \to \mathbf{C}$ such that $\Sigma|f(n)|^2\beta(n)$ is finite, and let S_j denote the 'backwards shift' operator

$$S_j f(n) = f(n + e_j)$$

on this space. Here e_j is the multi-index given by $(e_j)_k = \delta_{jk}$ (Kronecker's delta). Clearly the S_j commute. Routine calculations show that

$$(S_j^* g)(n) = g(n - e_j)\,\beta(n - e_j)(\beta(n))^{-1} = g(n - e_j)n_j|n|^{-1}. \tag{4}$$

In the case $n_j = 0$ the second and third expressions in (4) are interpreted as zero. Then $I - S^* \cdot S$ is the projection operator on the 1-dimensional subspace of elements carried by the zero multi-index. In particular, (2) holds.

2. Proof of the Theorem. The proof given here is based on Nagy and Foiaş [**4**, Chapter I, §10]. It will be enough to show that (3) holds when (2) is replaced by

$$[A_{j_1}, A_{j_2}] = 0 \quad (1 \leqslant j_1, j_2 \leqslant J) \quad \text{and} \quad \sum_{j=1}^{J} \|A_j h\|^2 \leqslant r^2\|h\|^2 \ \forall h \in H \tag{5}$$

for r satisfying $0 \leqslant r < 1$. Let r be fixed in this range and suppose that (5) holds. Then the positive definite defect operator D given by $D^*D = I - A^* \cdot A$ is invertible. Now define a new hilbert space $\check{H}$ to have the same underlying space as H and the norm

$$\|h\|_{\check{H}} = \|Dh\|_H.$$

We construct a 'dilation' on the space $\tilde{H} = l^2(\mathbf{Z}_+^J, \beta; \check{H})$. The inclusion map Θ: $H \to \tilde{H}$ is given by

$$\Theta h(n) = A^n h \quad \forall h \in H.$$

By elementary properties of the multinomial coefficients we have

$$\sum_{|n|=k+1} \beta(n)A^{*n}A^n = \sum_{|m|=k} \beta(m)A^{*m}(A^* \cdot A)A^m;$$

also one shows by induction on k that $\|\Sigma_{|n|=k}\beta(n)A^{*n}A^n\| \leqslant r^{2k}$; it follows that the sum

$$\sum_{k=0}^{\infty} \sum_{|n|=k} \beta(n)A^{*n}(I - A^* \cdot A)A^n$$

converges absolutely and 'telescopes' to the identity operator. Thus

$$\sum_n \beta(n)\|A^n h\|_{\check{H}}^2 = \sum_n \beta(n)\|DA^n h\|_H^2 = \|h\|_H^2$$

so that Θ is indeed an isometric inclusion. Next let $\tilde{S}_j$ denote the (vector-valued) backwards shift on $\tilde{H}$ analogous to S_j. Then the intertwining relation

$$\tilde{S}^n\Theta = \Theta A^n \quad \forall n \in \mathbf{Z}_+^J$$

holds. From this we may deduce that $\|Q(A)\| \leqslant \|Q(\tilde{S})\|$ for every complex polynomial Q of J variables. On the other hand, since $\tilde{S} = S \otimes I_{\check{H}}$, we have $Q(\tilde{S}) = Q(S) \otimes I_{\check{H}}$, and it follows that $\|Q(\tilde{S})\| \leqslant \|Q(S)\|$. Thus (3) holds as required.

3. Concluding remarks. In this section we relate $\|Q(S)\|$ to a certain multiplier norm. Let $\mathcal{K}$ denote the hilbert space of functions φ which admit a power series representation

$$\varphi(z) = \sum_n a(n)\, z^n \tag{6}$$

with $a(n) \in \mathbf{C}$ for $n \in \mathbf{Z}_+^J$ satisfying

$$\|\varphi\|_{\mathcal{K}}^2 = \sum_n (\beta(n))^{-1}|a(n)|^2 < \infty. \tag{7}$$

The norm condition (7) implies that the series (6) converges absolutely for $|z| < 1$ so that $\mathcal{K}$ may be viewed as a hilbert space of functions holomorphic in the open unit ball. Let M_j denote the multiplication operator on $\mathcal{K}$ defined by z_j and let M denote the corresponding J-tuple.

LEMMA. *$\|Q(M)\| = \|Q(S)\|$ for every polynomial Q.*

It is worth noting that M does not satisfy the second half of (2). The operator $Q(M)$ is, of course, just the multiplication operator on $\mathcal{K}$ defined by Q.

PROOF OF THE LEMMA. By virtue of (4) the isometric isomorphism T: $l^2(\mathbf{Z}_+^J, \beta) \to \mathcal{K}$ given by

$$(Tg)(z) = \sum g(n)\,\beta(n) z^n$$

satisfies the intertwining relation $TS_j^* = M_j T$ $(1 \leqslant j \leqslant J)$. Let Q be given and define a new polynomial $\bar{Q}$ by $\bar{Q}(z) = \overline{Q(\bar{z})}$. Then we have

$$\|Q(S)\| = \|Q(S)^*\| = \|\bar{Q}(S^*)\| = \|\bar{Q}(M)\|. \tag{8}$$

Also, since the isometric conjugation operator on $\mathcal{K}$ given by $\varphi \to \bar{\varphi}$ intertwines $Q(M)$ and $\bar{Q}(M)$ we have that $\|Q(M)\| = \|\bar{Q}(M)\|$. Combining this with (8) we have the conclusion of the Lemma.

COROLLARY. *Let A be a J-tuple satisfying* (2) *and let Q be a complex polynomial. Then $\|Q(A)\| \leqslant \|Q(M)\|$.*

In the case $J = 1$, $\mathcal{K}$ is just the Szegö space H^2 so that $\|Q(M)\| \leqslant \|Q\|_\infty$ (actually equality occurs). Thus the Corollary yields the original von Neumann inequality (1).

It is well known that for large J the direct generalization of von Neumann's inequality fails [2], [3]. In our context it fails for $J \geqslant 2$. Indeed in the

essentially typical case $J = 2$ we consider $Q_n(z) = (2z_1z_2)^n$. Then we have that $\|Q_n\|_\infty = \sup\{|Q_n(z)|;\ |z| \leqslant 1\} = 1$ whilst, since $\|\mathbf{1}\|_{\mathcal{K}} = 1$,

$$\|Q_n(M)\|^2 \geqslant \|Q_n\|_{\mathcal{K}}^2 = 4^n(n!)^2(2n!)^{-1},$$

which is unbounded.

Finally, we give a crude method for estimating the multiplier norm of $\mathcal{K}$. Let ∂B_J denote the unit sphere in $\mathbf{C}^J$ and σ the normalized area measure. Let $\mathcal{L}$ denote the hilbert space of functions $f(\lambda) = \sum_{k=0}^{\infty} a_k\lambda^k$ on the multiplicative circle group $\mathbf{T}$ satisfying

$$\|f\|_{\mathcal{L}}^2 = \sum_{k=0}^{\infty} |a_k|^2\binom{J+k-1}{k} < \infty.$$

For $\lambda \in \mathbf{T}$ and $z \in \mathbf{C}^J$ we denote by λz the element $(\lambda z_1, \ldots, \lambda z_J)$.

PROPOSITION. *Let Q be a polynomial of J variables, and for $z \in \partial B_J$ define a polynomial Q_z of one variable by $Q_z(\lambda) = Q(\lambda z)$. Then*

$$\|Q(M)\| \leqslant \sup_{z \in \partial B_J} \|Q_z\|_{\mathrm{mult}(\mathcal{L})}. \tag{9}$$

Here $\|P\|_{\mathrm{mult}(\mathcal{L})}$ denotes the norm of P as a multiplier of the space $\mathcal{L}$.

The significance of the Proposition is that the right-hand side of (9) takes account of the smoothness of Q on ∂B_J only in the direction of "i times the normal".

PROOF OF THE PROPOSITION. It is well known [5] that the functions of the Szegö space $H^2(\partial B_J, \sigma)$ are those that admit a power series expansion $\theta(z) = \sum_n a(n)z^n$, where $\|\theta\|_2^2 = \sum_n(\alpha(n))^{-1}|a(n)|^2 < \infty$. The mapping N: $\mathcal{K} \to H^2(\partial B_J, \sigma; \mathcal{L})$, defined by

$$(N\mathcal{X})(z, \lambda) = \mathcal{X}(\lambda z) \quad \forall \mathcal{X} \in \mathcal{K}, \lambda \in \mathbf{T}, |z| < 1,$$

is an isometry, for if $\mathcal{X}(z) = \sum a(n)z^n$, then $N\mathcal{X}(z, \lambda) = \sum a(n)z^n\lambda^{|n|}$, and we have that

$$\begin{aligned}\|N\mathcal{X}\|^2 &= \sum_n (\alpha(n))^{-1}\binom{J+|n|-1}{|n|}|a(n)|^2 \\ &= \sum_n (\beta(n))^{-1}|a(n)|^2 = \|\mathcal{X}\|_{\mathcal{K}}^2.\end{aligned}$$

Now let $\mathcal{X}$ be an element of $\mathcal{K}$ and assume that the right-hand side of (9) is bounded by 1. Then we have that

$$(N(Q\mathcal{X}))(z, \lambda) = Q_z(\lambda)(N\mathcal{X})(z, \lambda)$$

and that

$$\|(N(Q\mathcal{X}))(z, \cdot)\|_{\mathcal{L}} \leqslant \|(N\mathcal{X})(z, \cdot)\|_{\mathcal{L}}. \tag{10}$$

Squaring (10), integrating with respect to the variable z and using the fact that N is an isometry now yields $\|Q\mathcal{X}\|_{\mathcal{K}} \leqslant \|\mathcal{X}\|_{\mathcal{K}}$. Since $\mathcal{X}$ was an arbitrary element of $\mathcal{K}$ this yields the desired result.

REFERENCES

1. J. von Neumann, *Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes*, Math. Nachr. **4** (1951), 258–281.

2. M. J. Crabb and A. M. Davie, *Von Neumann's inequality for Hilbert space operators*, Bull. London Math. Soc. **7** (1975), 49–50.

3. N. Th. Varopoulos, *On an inequlaity of von Neumann and an application of the metric theory of tensor products to operator theory*, J. Functional Analysis **16** (1974), 83–100.

4. B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*, North-Holland, Amsterdam, 1970.

5. C. F. Dunkl and D. E. Ramirez, *Topics in harmonic analysis*, Appleton Century Crofts, New York, 1971.

DEPARTMENT OF MATHEMATICS, MCGILL UNIVERSITY, MONTREAL H3A 2K6, CANADA