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ON GENERICITY AND COMPLEMENTS OF MEASURE ZERO SETS IN FUNCTION SPACES

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ABSTRACT. Generic properties of function spaces have been of particular interest in dynamical systems and singularity theory. The underlying assumption has been that the complement of a dense G_δ set is sparse enough to be considered unlikely. Nevertheless, in infinite dimensional spaces, even dense G_δ's may have measure zero. Since there is no one canonical measure on an infinite dimensional Fréchet space, notions of measure zero have not often been considered. Here we use a notion of Haar measure zero on abelian Polish groups due to Christensen [1]. We show that those sections of a finite dimensional vector bundle over a compact manifold whose jets are transverse to a submanifold of the jet bundle are complements of sets of Haar measure zero.

An abelian Polish group G is an abelian topological group with group operation $+$ such that G has a separable Hausdorff topology and such that there exists at least one complete metric on d inducing the given topology.

A universally measurable subset of G is a subset of G that is measurable for every probability measure defined on the Borel sets of G.

We will say that a universally measurable subset A of G has Haar measure zero if there is a (nonunique) probability measure du on G, called a testing measure, such that for any $g \in G$, $\int_G \chi_{A+g}\, du = 0$. Here $A + g$ is the g-translate of A, and for any $S \subset G$, χ_s is the characteristic function of S. That is, the set A has Haar measure zero if A and all its translates have measure zero with respect to the testing measure.

We observe without proof that if G were a locally compact group with Haar measure dh, then the notion we have defined is equivalent to saying that $\int_G \chi_A\, dh = 0$. We also observe that the countable union of Haar measure zero sets is a set of Haar measure zero. The justification for interest in this notion of measure zero is that it seems to be a suitable one for doing calculus on abelian Polish groups. In particular, there is a generalization of a theorem of Rademacher saying that Lipschitz mappings between certain abelian Polish groups have directional derivatives linear in the direction a.e. in this sense.

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See [1] for details and further references.

Let M be a compact, connected C^∞ manifold. Let B be a vector bundle over M with finite dimensional fiber. We will use the symbol $C^\infty(B)$ to denote the set of C^∞ sections of B with addition of sections as the group operation. For any nonnegative integer k, we will let $J^k(B)$ denote the k-jet bundle of sections from M to B. There is a well-known transversality theorem that says that if W is a submanifold of $J^k(B)$, then there is a residual subset R of G such that if $g \in R$ then $j^k g$: $M \to J^k(B)$ is transversal to W (see [2, II. 4.9]). Here $j^k g$ is the k-jet mapping of g. Our main result will be

I. THEOREM. *Let $G = C^\infty(B)$. Let W be a submanifold of $J^k(B)$ and let R be the subset of those sections $g \in G$ such that $j^k(g)$: $M \to J^k(B)$ is transverse to W. Then G is an abelian Polish group and R is the complement of a set of Haar measure zero in G.*

To prove the theorem, we will need some lemmas.

II. LEMMA. *G is separable.*

PROOF OF LEMMA II. Since M is compact, we can find a positive integer p and for each $i = 1, 2, \ldots, p$ we can find a C^∞ function ϕ_i: $M \to [0, 1]$ such that

1. (i) ϕ_i has compact support C_i,

(ii) $C_i \subset U_i \subset M$, where U_i is open and diffeomorphic to an open subset of $\mathbf{R}^m$, and

(iii) $\sum_{i=1}^p \phi_i(x) = 1$ for any $x \in M$.

As a result, any section $g \in G$ can be written as a finite sum $g = \sum \phi_i g$. Here $\phi_i g$ is a C^∞ section of B with compact support in C_i.

We will show the existence of a countable set of sections that are dense in the set $G_i = \{\phi_i g\colon g \in G\}$. This will complete the proof of Lemma II.

We may, without loss of generality, assume that the restriction of B to U_i is trivial. That is, $B|_{U_i} \approx U_i \times \mathbf{R}^n$ for some integer n. In particular, we may identify U_i with an open subset of $\mathbf{R}^m$ and identify sections of $B|_{U_i}$ with C^∞ maps from U_i to $\mathbf{R}^n$. By the Weierstrass approximation theorem, see [3], those maps whose component functions are polynomials with rational coefficients can be used to uniformly approximate any other map up to order k in the C^k sup norm on C_i. In particular, if we denote those countably many polynomial maps by $\{p_j\colon j \in N\}$, then $\{\phi_i p_j\colon j \in N\}$ is dense in G_i. This proves the lemma.

II. LEMMA. *There is a complete metric d on G that induces the Whitney C^∞ topology.*

PROOF. The Whitney C^∞ topology is the coarsest topology which is finer than the Whitney C^k topology for any finite k. If $G^k = C^k(B)$ is the set of C^k sections of B in the Whitney C^k topology, then there is a complete metric d^k on G^k (see [2, p. 43]). Since $G \subset G^k$, there is an induced metric on G which we will also call d^k. If $g, g' \in G$, we will define a metric on G by

$$d(g, g') = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{d^k(g,g')}{1 + d^k(g,g')}.$$

It is not hard to see that d is complete and induces the proper topology on G.

Now let us prove Theorem I. As in [2, II. 4.9], we choose a countable open cover of W by open sets $W_1, W_2, \ldots$ such that:

2. (i) the closure of W_i in $J^k(B)$ is contained in W;

(ii) $\overline{W}_i$ is compact;

(iii) there is a chart U_i on M such that $B|_{U_i} \approx U_i \times \mathbf{R}^n$ for some integer n and such that $\pi(\overline{W}_i) \subset U_i \times \mathbf{R}^n$, where $\pi\colon J^k(B) \to B$ is the natural projection mapping; and

(iv) $\overline{U}_i$ is compact.

If we let $T_i = \{g \in G\colon j^k g \pitchfork W \text{ on } \pi\overline{W}_i\}$, then just as in [2], each T_i is open and dense and $R = \bigcap T_i$. Let us show that for each i, the set CT_i, the complement of T_i, is a set of Haar measure zero.

Let π_1 be the projection of $J^k(B)$ onto M. Note that $\overline{W}_i \subset \pi_1^{-1}(U_i)$. We can always assume that we have chosen the W_i and U_i small enough so that there exist a finite integer q and q sections $g_1, g_2, \ldots, g_q \in G$ such that:

3. (i) for each $g \in G$ we can define a map $\bar{g}\colon \mathbf{R}^q \to G$ by $\bar{g}(s_1, \ldots, s_q)(x) = (s_1 g_1 + \cdots + s_q g_q + g)(x)$;

(ii) the map $\underline{g}$ from $\mathbf{R}^q \times M \to J^k(B)$, defined by $\underline{g}((s_1, \ldots, s_q), x) = j^k(\bar{g}(s_1, \ldots, s_q))(x)$, is onto $\pi_1^{-1}(U_i)$;

(iii) for each fixed $x \in U_i$, the map $\underline{g}((\cdots), x)$ from $\mathbf{R}^q$ to $J^k(B)$ is surjective at each point in its domain.

To see that such $g_1, \ldots, g_q$ can exist, we identify U_i with a neighborhood of zero in $\mathbf{R}^m$ and let $g_1, \ldots, g_q$ be the set of all distinct maps to $\mathbf{R}^n$ with only one nonzero entry that is a monomial of order $\leqslant k$ in m variables.

Let $S = \bar{0}(\mathbf{R}^q)$ be the image of $\mathbf{R}^q$ in G. Here 0 is the zero section of B. We will induce a probability measure on S by using a probability measure on $\mathbf{R}^q$ which assigns probability zero to sets of Lebesgue measure zero. If $R \subset S$, we define the measure of R to be the measure of $(\bar{0})^{-1}(R)$. If $R \subset G$, but R is not a subset of S, we define the measure of R to be the measure of $R \cap S$.

We wish to show that CT_i has measure zero with respect to the testing measure we have just defined.

By hypotheses 3(i)–(iii), for any $g \in G$, the map $\underline{g}\colon \mathbf{R}^q \times M \to J^k(B)$ is transversal to W on U_i. By Lemma II.4.6 of [2], the set A_g of points $(s_1, \ldots, s_q) \in \mathbf{R}^q$ such that $\underline{g}((s_1, \ldots, s_q), \cdot)$ is transversal to W on U_i is open and dense in $\mathbf{R}^q$. Careful reading of the proof of the lemma shows that $\mathbf{R}^q - A_g$ is a set of measure zero.

In particular, $\mathbf{R}^q - A_0 = (\bar{0})^{-1}(CT_i \cap S)$ and $\mathbf{R}^q - A_{-g} = \bar{0}^{-1}[(CT_i + g) \cap S] = (-\bar{g})^{-1}(CT_i \cap S)$. That is CT_i and all its translates have measure zero. Thus CT_i must be a set of Haar measure zero.

REMARKS. 1. The same proof shows that if W is a submanifold of $J^k(B)$, then the elements of G^{k+1} that are transversal to W are the complement of a set of Haar measure zero. Recall that G^{k+1} is the set of C^{k+1} sections of B in the Whitney C^{k+1} topology.

2. If we are considering the set of C^∞ maps from compact connected M to N in the C^∞ topology, these form a Fréchet manifold (see [2, III.1.11]). Given $f \in C^\infty(M, N)$, we can find a neighborhood U of f in $C^\infty(M, N)$ that can be identified with an open subset of a Fréchet space and show that those elements of U transversal to W on M are the complement of a set of Haar measure zero.

3. Even if M is noncompact, at first it seems that the definition of Haar measure zero still applies to sections of $C^\infty(B)$ where B is a vector bundle over M. In this case, Christensen's proof that countable union of measure zero sets has measure zero does not hold. Also, if M is noncompact, $C^\infty(B)$ is not separable and the generalization of Rademacher's theorem may not hold.

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