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WEAK CONVERGENCE TO THE FIXED POINT OF AN ASYMPTOTICALLY NONEXPANSIVE MAP

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ABSTRACT. It is proved that, in certain Banach spaces, any asymptotically nonexpansive and asymptotically regular map has the property that its iterates, applied to any point in the domain, give a sequence converging weakly to a fixed point.

Our object is to extend Opial's convergence theorem (Theorem 2 in [7]) to the case of asymptotically nonexpansive map. Suppose K is a nonempty bounded closed convex subset of a Banach space X. A mapping T: $K \to K$ is called asymptotically nonexpansive [5] if for each $x, y \in K$

$$(*) \qquad \|T^i x - T^i y\| \leqslant k_i \|x - y\|, \qquad i = 1, 2, \dots,$$

where $\{k_i\}$ is a fixed sequence of positive reals such that $k_i \to 1$ as $i \to \infty$. Existence of fixed points of such a mapping when X is uniformly convex has been proved by Goebel and Kirk in [5].

In §1, we recall some basic definitions and known results. In §2, after Kirk we construct what we call the asymptotically central set of a sequence and observe some simple facts about it. Our main results are contained in §3.

1. A mapping T: $K \to K$ is called nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for any x, y in K. It is called asymptotically nonexpansive if it satisfies $(*)$ above. T is asymptotically regular if for any $x \in K$, $T^n x - T^{n+1} x \to 0$, as $n \to \infty$. It is demiclosed if for any sequence $\{x_n\} \subset K$, $x_n \rightharpoonup x_0$ and $Tx_n \to y_0$ $\Rightarrow Tx_0 = y_0$ where $\rightharpoonup$ denotes weak convergence. The modulus of convexity of X is a function δ: $[0, 2] \to [0, 1]$ defined by $\delta(\varepsilon) = \inf\{1 - \frac{1}{2}\|x + y\|: \|x\| \leqslant 1, \|y\| \leqslant 1, \|x - y\| \geqslant \varepsilon\}$. It is known that δ is a nondecreasing function and is continuous on $[0, 2)$. It is also known [8], [9] that

$$(**) \qquad \begin{gathered} \|x\| \leqslant d, \quad \|y\| \leqslant d, \\ \|x - y\| \geqslant \varepsilon \Rightarrow \tfrac{1}{2}\|x + y\| \leqslant \left(1 - \delta\left(\frac{\varepsilon}{d}\right)\right) d. \end{gathered}$$

Opial [7] has shown that in a uniformly convex Banach space having weakly continuous duality mapping (or in a Hilbert space) if a sequence $\{x_n\}$ converges weakly to an x_0 then

$$(0) \qquad \liminf_n \|x_n - x_0\| < \liminf_n \|x_n - x\| \quad \forall x \neq x_0.$$

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It is easy to observe (see [2]) that the above inequality can be given an equivalent form in terms of lim sup:

$$(0') \qquad \limsup_n \|x_n - x_0\| < \limsup_n \|x_n - x\| \quad \forall x \neq x_0.$$

2. Let K be a nonempty bounded closed convex subset of a reflexive Banach space X and let $\{x_n\}$ be any sequence in K. Following Kirk [6] and Edelstein [4] let us define the following:

$$r(x) = \limsup_n \|x_n - x\|, \qquad x \in X.$$

This r is a continuous function of X into the reals (see Edelstein [3]).

$$\rho = \rho_K(\{x_n\}) = \inf\{r(x): x \in K\},$$
$$C_0 = \{x \in K: r(x) = \rho\}.$$

ρ is called the asymptotic radius of $\{x_n\}$ in K and we prefer to call C_0 the asymptotically central set of $\{x_n\}$ in K. C_0 is a singleton if the space is uniformly convex (Proposition 2 below). In that case it is called asymptotic center.

Let $B_n(r)$ denote the closed ball of radius r centered at x_n and define

$$C_\varepsilon = \bigcup_{i \geqslant 1} \left(\bigcap_{n \geqslant i} B_n(\rho + \varepsilon) \right).$$

PROPOSITION 1 [5]. *$C_0 = \bigcap_{\varepsilon>0}(K \cap \overline{C}_\varepsilon)$ and is a nonempty closed convex subset of K.*

PROPOSITION 2 [3]. *If the space is uniformly convex then C_0 is a singleton.*

The following lemma easily follows from Proposition 2 and the inequality (0′).

LEMMA 1. *Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space having weakly continuous duality mapping. If a sequence $\{x_n\} \subset K$ converges weakly to a point x_0 then x_0 is the asymptotic center of $\{x_n\}$ in K.*

In the proof of Theorem 2 in [1] we have observed that the space being uniformly convex if T: $K \to K$ is asymptotically nonexpansive, then the asymptotic center of $\{T^n x\}$ in K for any $x \in K$ is a fixed point of T. We now prove:

3. LEMMA 2. *Let K and X be as in Lemma 1, T: $K \to K$ an asymptotically nonexpansive mapping. Suppose x_0 is the asymptotic center of the sequence $\{T^n x\}$ for some $x \in K$. If the weak limit ξ_0 of a subsequence $\{T^{n_i} x\}$ is a fixed point of T, then it must coincide with x_0 (which is a fixed point as remarked above).*

PROOF. Let ρ and ρ' be the asymptotic radii respectively of $\{T^n x\}$ and $\{T^{n_i} x\}$. Clearly $\rho' \leqslant \rho$. Since $\{T^{n_i} x\}$ converges weakly to ξ_0, by Lemma 1, ξ_0

must be the asymptotic center of $\{T^{n_i}x\}$ in K, so that given any $\varepsilon > 0$ we can choose an integer i_0 such that

$$\|\xi_0 - T^{n_{i_0}}x\| \leqslant \rho' + \varepsilon/2.$$

Since ξ_0 is a fixed point of T and T is asymptotically nonexpansive we can choose an integer J such that

$$\left\|\xi_0 - T^{n_{i_0}+j}x\right\| \leqslant k_j(\rho' + \varepsilon/2) \leqslant \rho' + \varepsilon \leqslant \rho + \varepsilon \quad \text{for all } j \geqslant J.$$

It follows therefore that $\lim\sup_n \|\xi_0 - T^n x\| = \rho$ and x_0 being the unique point with this property we have $\xi_0 = x_0$. The proof is complete.

Now we prove our convergence theorem.

THEOREM. *Let X be a uniformly convex Banach space having weakly continuous duality mapping and K a nonempty bounded closed convex subset of X. Suppose T: $K \to K$ is asympotically nonexpansive and asymptotically regular. Then for any $x \in K$, the sequence $\{T^n x\}$ converges weakly to a fixed point of T.*

PROOF. We will show that the asymptotic regularity of T makes every weak cluster point of $\{T^n x\}$ a fixed point of T. In view of the above lemma this would mean that all the weak cluster points of $\{T^n x\}$ coincide with the asymptotic center x_0 of $\{T^n x\}$ in K (which is a fixed point) and would complete the proof.

Let us suppose that the subsequence $\{T^{n_i}x\}$ converges weakly to ξ_0. Then by Lemma 1, ξ_0 will be the asymptotic center of $\{T^{n_i}x\}$ in K, let the asymptotic radius be ρ. By asymptotic regularity of T, $(I - T)\, T^{n_i}x \to 0$ as $i \to \infty$. Since $\{T^{n_i}x\}$ converges weakly to ξ_0, this implies $\{T^{n_i+1}x\}_{i=1}^{\infty}$ converges weakly to ξ_0. It follows in the same way that for any integer r the sequence $\{T^{n_i+r}x\}_{i=1}^{\infty}$ converges weakly to ξ_0 and thus all these sequences have the same asymptotic center ξ_0 in K. We now claim that all these sequences have the same asymptotic radius ρ.

We have

$$\begin{aligned}\|\xi_0 - T^{n_i+1}x\| - \|\xi_0 - T^{n_i}x\| &\leqslant \left\|(\xi_0 - T^{n_i+1}x) - (\xi_0 - T^{n_i}x)\right\| \\ &= \|T^{n_i}x - T^{n_i+1}x\| \to 0 \quad \text{as } i \to \infty\end{aligned}$$

by asymptotic regularity of T. Thus

$$\lim\sup_i \|\xi_0 - T^{n_i+1}x\| = \lim\sup_i \|\xi_0 - T^{n_i}x\| = \rho$$

and our claim follows.

We now prove that ξ_0 is a fixed point of T. If we can show that $T^j\xi_0 \to \xi_0$ as $j \to \infty$, by continuity of T this will mean ξ_0 is a fixed point of T, so let us suppose $T^j\xi_0$ does not converge to ξ_0. Then there is a $d > 0$ and a sequence $\{j_m\}$ of integers such that

$$\|\xi_0 - T^{j_m}\xi_0\| \geqslant d \quad \text{for all } m.$$

By uniform convexity of the space, we may choose an $\varepsilon > 0$ such that

$(\rho+\varepsilon)[1-\delta(d/(\rho+\varepsilon))]<\rho$. Since all the sequences

$$\{T^{n_i+r}x\}_{i=1}^{\infty},\quad r=0,1,2,\ldots,$$

have the same asymptotic center ξ_0 and the same asymptotic radius ρ, there exist integers $I=I(r)$ such that

$$(1)\qquad \|\xi_0-T^{n_i+r}x\|\leqslant\rho+\varepsilon/2\quad\text{for all } i\geqslant I(r).$$

We have for any m,

$$\|T^{j_m}\xi_0-T^{n_i+j_m}x\|\leqslant k_{j_m}\|\xi_0-T^{n_i}x\|\leqslant k_{j_m}(\rho+\varepsilon/2)\quad\text{for } i\geqslant I(0).$$

We choose an integer M such that (as $k_j\to 1$ as $j\to\infty$) $k_{j_m}(\rho+\varepsilon/2)\leqslant\rho+\varepsilon$ for all $m\geqslant M$, so that we have

$$(2)\qquad \|T^{j_m}\xi_0-T^{n_i+j_m}x\|\leqslant\rho+\varepsilon\quad\text{for all } i\geqslant I(0)\text{ and all } m\geqslant M$$

and from (1) we have

$$(3)\qquad \|\xi_0-T^{n_i+j_m}x\|\leqslant\rho+\varepsilon\quad\text{for } i\geqslant I(j_m).$$

Since $\|\xi_0-T^{j_M}\xi_0\|\geqslant d$, (2) and (3) yield

$$\left\|\frac{\xi_0+T^{j_M}\xi_0}{2}-T^{n_i+j_M}(x)\right\|\leqslant(\rho+\varepsilon)\left[1-\delta\left(\frac{d}{\rho+\varepsilon}\right)\right]<\rho$$

for all $i\geqslant\max\{I(0),I(j_M)\}$. This is a contradiction in view of the fact that the sequence $\{T^{n_i+j_M}x\}_{i=1}^{\infty}$ has asymptotic radius ρ in K. The proof of the theorem is therefore complete.

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REFERENCES

1. S. C. Bose, *On nonexpansive and asymptotically nonexpansive mappings* (unpublished work).

2. E. Lami Dozo, *Multivalued nonexpansive mappings and Opial's condition*, Proc. Amer. Math. Soc. **38** (1973), 286–292.

3. M. Edelstein, *Fixed point theorems in uniformly convex Banach spaces*, Proc. Amer. Math. Soc. **44** (1974), 369–374.

4. ______, *The construction of an asymptotic center with a fixed point property*, Bull. Amer. Math. Soc. **78** (1972), 206–208.

5. K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35** (1972), 171–174.

6. W. A. Kirk, *On nonlinear mappings of strongly semicontractive type*, J. Math. Anal. Appl. **27** (1969), 409–412.

7. Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.

8. ______, *Lecture notes on nonexpansive and monotone mappings in Banach spaces*, Center for Dynamical Systems, Brown University, Providence, R. I., 1967.

9. H. Schaefer, *Uber die Methode sukzessiver Approximationen*, Jber. Deutsch. Math.-Verein. **59** (1957), 131–140.

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