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A COUNTEREXAMPLE IN THE FACTORIZATION OF BANACH SPACE OPERATORS

RICHARD BOULDIN

ABSTRACT. A counterexample is given which completes the Banach space generalization of a theorem of R. G. Douglas concerning the factorization of Hilbert space operators.

In [1] R. G. Douglas proves the equivalence of three conditions related to factoring a Hilbert space operator. In [2] Mary Embry determines all but one of the possible implications among those three conditions interpreted for operators on a Banach space. This short note gives a counterexample to show that the one remaining possible implication does not hold.

In the notation of [4] define A and B on (c_0) by $Ae_k = 0$ for $k \neq 1$ and $Ae_1 = y = (2^{-1}, 2^{-2}, \ldots)$ and $B(x_1, x_2, \ldots) = (2^{-1}x_1, 2^{-2}x_2, \ldots)$. Recall $(c_0)' = l^1$ and $(l^1)' = l^\infty$. The straightforward proof of the next lemma is omitted.

LEMMA. *The dual operator B'' on l^∞ maps $(f_1, f_2, \ldots)$ to $(2^{-1}f_1, 2^{-2}f_2, \ldots)$ and the images of B and B'' are $\{(x_1, x_2, \ldots) \in (c_0)\colon \lim 2^n x_n = 0\}$ and $\{(h_1, h_2, \ldots) \in l^\infty\colon \sup|2^n h_n| < \infty\}$, respectively. The operator A'' on l^∞ is defined by $A''(g_1, g_2, \ldots) = g_1 h$, where $h = (2^{-1}, 2^{-2}, \ldots)$ and the images of A and A'' are* $\operatorname{span}\{y\}$ *and* $\operatorname{span}\{h\}$, *respectively.*

THEOREM. *Condition* (i) *below holds but* (ii) *does not hold*:

(i) *for some $c \geqslant 0$, $\|A'f\| \leqslant c\|B'f\|$ for all $f \in l^1$*;

(ii) *the image of B contains the image of A–i.e. $B(c_0) \supset A(c_0)$.*

PROOF. From the Lemma it follows that $A''l^\infty \subset B''l^\infty$ and $A(c_0) \not\subset B(c_0)$. Theorem 1 of [2] implies that (i) above holds.

This counterexample fills a gap in [3] showing that the converse to Theorem 1 of [3] does not hold.

REFERENCES

[1] R. G. Douglas, *On majorization, factorization and range inclusion of operators in Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413–416.

[2] Mary Embry, *Factorization of operators on Banach space*, Proc. Amer. Math. Soc. **38** (1973), 587–590.

[3] Robin Harte, *Berberian-Quigley and the ghost of a spectral mapping theorem*, Proc. Roy. Irish Acad. (to appear).

[4] A. E. Taylor, *Introduction to functional analysis*, Wiley, New York, 1963.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GEORGIA, ATHENS, GEORGIA 30602

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