PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 69, Number 2, May 1978

A SHORT PROOF OF A GREENE THEOREM

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ABSTRACT. A short and simple proof of an inequality of the Gronwall type is given for a class of integral systems based upon the generalized Gronwall lemma of Sansone-Conti.

Recently David E. Greene [1] used a technically involved iteration in proving the following

THEOREM [GREENE]. *Let K_1, K_2 and μ be nonnegative constants and let f, g and h_i be continuous nonnegative functions for all $t \geqslant 0$ with h_i bounded such that*

$$f(t) \leqslant K_1 + \int_0^t h_1(s) f(s)\, ds + \int_0^t e^{\mu s} h_2(s) g(s)\, ds,$$

$$g(t) \leqslant K_2 + \int_0^t e^{-\mu s} h_3(s) f(s)\, ds + \int_0^t h_4(s) g(s)\, ds$$

for all $t \geqslant 0$. Then there exist constants c_i and M_i such that

$$f(t) \leqslant M_1 e^{c_1 t}, \qquad g(t) \leqslant M_2 e^{c_2 t}$$

for all $t \geqslant 0$.

In this note, is presented a short and simple proof of this theorem based upon the generalized Gronwall lemma of Sansone-Conti which is cited from [2] in a restricted form as follows

(GENERALIZED) GRONWALL LEMMA. *For all $t \geqslant 0$, let three functions λ, ϕ, u be given such that λ is summable and nonnegative, ϕ is absolutely continuous, and u is continuous. If $u(t) \leqslant \phi(t) + \int_0^t \lambda(s) u(s)\, ds$, then*

$$u(t) \leqslant \int_0^t \phi'(s) \exp\left(\int_s^t \lambda(r)\, dr\right) ds + \phi(0) \exp\left(\int_0^t \lambda(r)\, dr\right).$$

PROOF OF THE THEOREM. Let P be an upper bound for h_i (the assumption $\mu > 0$ is not necessarily required here), then

$$f(t) \leqslant K_1 + P\int_0^t f(s)\, ds + P\int_0^t e^{\mu s} g(s)\, ds, \tag{1}$$

$$g(t) \leqslant K_2 + P\int_0^t e^{-\mu s} f(s)\, ds + P\int_0^t g(s)\, ds. \tag{2}$$

Received by the editors July 5, 1977.

AMS (MOS) subject classifications (1970). Primary 34C10; Secondary 45F05.

[1]The author was supported (in part) by the N.R.C. of Canada (Grant No. A3116).

Multiplying (1) by $e^{-\mu t}$ and then adding to (2),

$$\begin{aligned} &e^{-\mu t}f(t) + g(t) \\ &\quad \leqslant K_1 e^{-\mu t} + K_2 + \int_0^t P\left[e^{\mu(s-t)} + 1\right]\left[e^{-\mu s}f(s) + g(s)\right] ds \\ &\quad \leqslant K_1 e^{-\mu t} + K_2 + \int_0^t 2P\left[e^{-\mu s}f(s) + g(s)\right] ds. \end{aligned} \tag{3}$$

Applying the lemma to (3)

$$\begin{aligned} &e^{-\mu t}f(t) + g(t) \\ &\quad \leqslant \int_0^t (-K_1 \mu e^{-\mu s})\exp\left(\int_s^t 2P dr\right) ds + (K_1 + K_2)\exp\left(\int_0^t 2P ds\right) \\ &\quad = \frac{K_1 \mu}{2P + \mu} e^{-\mu t} + \frac{2P(K_1 + K_2) + K_2 \mu}{2P + \mu} e^{2Pt}. \end{aligned}$$

The conclusion of the theorem is now clear.

References

1. David E. Greene, *An inequality for a class of integral systems*, Proc. Amer. Math. Soc. **62** (1977), 101–104.

2. G. Sansone and R. Conti, *Non-linear differential equations*, rev. ed., Internat. Ser. of Monographs in Pure and Appl. Math., Vol. 67, Macmillan, New York, 1964. MR **31** #1417.

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