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THE $SC_{k+1}P$-INTEGRAL AND TRIGONOMETRIC SERIES

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ABSTRACT. Recently P. S. Bullen and C. M. Lee defined a scale of symmetric Cesàro-Perron integrals, but left open the question of whether their $SC_{k+1}P$-integral solves the coefficient problem for (C, k) summable series. This paper gives an affirmative answer to that question under natural conditions.

1. Introduction. The problem of constructing an integral in terms of which the coefficients of a (C, k) summable trigonometric series may be represented has been solved in several ways (see, e.g., [2], [6]–[10], [12], and [13]). In the case $k = 0$, J. C. Burkill's SCP-integral [6] solves the representation problem, although the integration by parts formula which was used in the original solution has yet to be verified. Recently H. Burkill [4] obtained the result without recourse to an integration by parts formula, using instead convergence properties of a series which is the formal product of a trigonometrical series and $\cos px$ or $\sin px$.

Bullen and Lee [3] have introduced a scale of symmetric Cesàro-Perron integrals, but left open the question of whether their $SC_{k+1}P$-integral solves the coefficient problem for (C, k)-summable series.

This paper gives an affirmative answer to that question under certain assumptions.

Our main result (Theorem 1) is weaker than the result in James [12, Theorem 6.2], since it requires the (C, k) summability of both the trigonometrical series and its conjugate except on a countable set. The result is weaker too than that of Mukhopadhyay [13, Theorem 8.1] for the same reason and also because in the latter the original series is required only to be summable (C, k) a.e. and to have bounded Cesàro means except on a countable set.

Insofar as application to trigonometrical series is concerned, the $SC_{k+1}P$-integral is thus seen to have the power of an "unsymmetric" rather than a "symmetric", integral (cf. [6] and [8]). This is related to the fact that in the formulation of the $SC_{k+1}P$-integral, the definition of $SC_{k+1}DF(x)$ is given in terms of the C_kP-integral (rather than the SC_kP-integral) of $F(x)$ because of the absence of an integration by parts formula for the SCP-integral.

2. Definitions. The notation and theory of the C_kP-integral [5] is assumed. For the theory of the SC_kP-integral we refer the reader to [3], but for

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convenience we reproduce the definition here.

Let F be a $C_{k-1}P$-integrable function on $[a, b]$, $k \geqslant 1$, and define

$$\Delta_k(F; x, h) = \frac{k+1}{2h}\{C_k(F; x, x+h) - C_k(F; x, x-h)\},$$

and

$$SC_kD_*F(x) = \liminf_{h\to 0} \Delta_k(F; x, h),$$

where $x \in (a, b)$ and $C_k(F; x, x+h)$ is the kth Cesàro mean of f in $(x, x+h)$ as defined in [5]. Define SC_kD^*F and SC_kDF in the usual way. The function F is said to be SC_k-continuous at x if $\lim_{h\to 0+} h\Delta_k(F; x, h) = 0$. It is easily seen that F is SC_k-continuous at x whenever it is C_k-continuous at x, and that $SC_kDF(x)$ exists and equals $C_kDF(x)$ whenever the latter exists.

Now suppose that f is a function defined and finite almost everywhere in $[a, b]$ and that B is a subset of $[a, b]$ of measure $b - a$, $a, b \in B$. The $C_{k-1}P$-integrable functions M and m are SC_kP-major and minor functions, respectively, of f on $[a, b]$ with base B if:

(2.1) M and m are SC_k-continuous on $[a, b]$ and C_k-continuous on B;

(2.2) $SC_kD_*M(x) \geqslant f(x) \geqslant SC_kD^*m(x)$, a.e. in (a, b);

(2.3) $SC_kD_*M(x) > -\infty$ and $SC_kD^*m(x) < +\infty$ except perhaps in a scattered set;

(2.4) $M(a) = 0 = m(a)$.

If f has SC_kP-major and minor functions and if

$$I \equiv \inf M(b) = \sup m(b) \neq \pm\infty,$$

then f is SC_kP-integrable on $[a, b]$ with base B and we write

$$I = SC_kP \int_{[a,b]}^{B} f(t)\, dt.$$

Suppose that F is a function defined on the interval $[a, b]$. If for $x_0 \in [a, b]$ there exist constants $\alpha_1, \alpha_2, \ldots, \alpha_r$ which depend on x_0 only and not on h, such that

$$F(x_0 + h) - F(x_0) = \sum_{k=1}^{r} \alpha_k \frac{h^k}{k!} + o(h^r), \quad \text{as } h \to 0,$$

then α_k, $1 \leqslant k \leqslant r$, is called the *Peano derivative of order* k of F at x_0 and is denoted by $F_{(k)}(x_0)$. If there exist constants $\beta_0, \beta_2, \ldots, \beta_{2r}$ which depend on x_0 only and not on h, such that

$$\frac{F(x_0 + h) + F(x_0 - h)}{2} = \sum_{k=0}^{r} \beta_{2k} \frac{h^{2k}}{(2k)!} + o(h^{2r}), \quad \text{as } h \to 0,$$

then β_{2k}, $0 \leqslant k \leqslant r$, is called the *de la Vallée Poussin derivative of order* $2k$ of F at x_0 and is denoted by $D^{2k}F(x_0)$. If F has derivatives $D^{2k}F(x_0)$, $0 \leqslant k \leqslant r - 1$, we write

$$\frac{h^{2r}}{(2r)!}\, \theta_{2r}(F; x_0, h) = \frac{F(x_0 + h) + F(x_0 - h)}{2} - \sum_{k=0}^{r-1} \frac{h^{2k}}{(2k)!} D^{2k}F(x_0),$$

and define

$$\overline{D}^{2r}F(x_0) = \limsup_{h\to 0} \theta_{2r}(F; x_0, h)$$

and

$$\underline{D}^{2r}F(x_0) = \liminf_{h\to 0} \theta_{2r}(F; x_0, h).$$

The de la Vallée Poussin derivatives are defined similarly for odd-numbered indices (see, e.g., [**11**, pp. 163–164]).

If $F_{(r)}(x_0)$ exists, so does $D^rF(x_0)$ and $F_{(r)}(x_0) = D^rF(x_0)$.

3. The representation of trigonometrical series in Fourier form. The series

$$\frac{1}{2}\, a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n(x) \tag{3.1.1}$$

is said to satisfy condition A if

(i) (3.1.1) and the series conjugate to (3.1.1), namely

$$\sum (b_n \cos nx - a_n \sin nx) \equiv \sum b_n(x), \tag{3.2}$$

are both summable (C, k) on $[0, 2\pi] - E \equiv B$, where E is a countable subset of $[0, 2\pi]$, and

(ii) $A_n^{k-1}(x) = o(n^k)$, $B_n^{k-1}(x) = o(n^k)$ for $x \in E$, where $A_n^{k-1}(x)$ and $B_n^{k-1}(x)$ denote the Cesàro means of order $k - 1$ of series (3.1.1) and (3.2), respectively, at x.

Clearly (i) implies $a_n = o(n^k)$, $b_n = o(n^k)$, $A_n^{k-1}(x) = o(n^k)$ and $B_n^{k-1}(x) = o(n^k)$, $x \in [0, 2\pi] - E$.

If (3.1.1) satisfies condition A we consider the following array of series:

$$\frac{a_0}{2} + \sum a_n(x) \equiv f(x), \quad (C, k), \qquad x \in [0, 2\pi] - E,$$

$$\frac{a_0 x}{2} - \sum \frac{b_n(x)}{n} \equiv F^{k+1}(x), \quad (C, k-1), \qquad x \in [0, 2\pi] - E, \tag{3.1.2}$$

$$\frac{a_0 x^2}{2 \cdot 2} - \sum \frac{a_n(x)}{n^2} \equiv F^k(x), (C, k-2), \qquad x \in [0, 2\pi] - E, \tag{3.1.3}$$

$$\cdots$$

where the convergence of series $(3.1.k + 3)$ is uniform to a continuous function F.

It follows from Theorem 3.1 [**11**] that F is $(k + 2)$ smooth everywhere, and, at points of B,

$$\frac{a_0 x^{2r}}{2 \cdot (2r)!} + (-1)^r \sum_{n=1}^{\infty} \frac{a_n(x)}{n^{2r}} = D^{k+2-2r}F(x), \quad (C, k-2r),$$

for $0 \leqslant r \leqslant ((k + 1)/2)$. Also by Theorem 3.2 [**12**] $F_r(x)$ exists for $0 \leqslant r \leqslant k$ and $x \in [0, 2\pi]$ (and equals $D_{(r)}F(x)$).

Since $F(x)$ can also be obtained by integrating (3.1.2) formally term by term $(k+1)$ times, it follows from Theorem 3.1 [**12**] again that $D^{k+1}F(x)$ exists and equals $F^{k+1}(x)$ on B. It follows easily that $D^{k+1}F(x) = F_{(k+1)}(x) = F^{k+1}(x)$ on B. It may be shown similarly that $F_{(r)}(x) = F^r(x)$, $x \in B$, $2 \leqslant r \leqslant k$. Finally it follows from a lemma of S. Verblunsky [**14**, Lemma 5] that $F_{(1)}(x) = F^1(x)$.

LEMMA 1. *If* (3.1.1) *satisfies condition* A *then* $F^{r+1}(x)$ *is* C_rP*-integrable on* $[0, 2\pi]$ *and*

$$F_{(r)}(t)\Big|_0^x = C_rP\int_0^x F^{r+1}(t)\,dt = C_rP\int_0^x F_{(r+1)}(t)\,dt, \tag{3.3}$$

$0 \leqslant r \leqslant k$, $0 \leqslant x \leqslant 2\pi$, *where* $F_{(0)}(x) = F(x)$.

PROOF. We have for each r, $0 \leqslant r \leqslant k$, $F_{(r)}(x)$ exists everywhere in $[0, 2\pi]$ and $F_{(r+1)}(x) = F^{r+1}(x)$ except possibly on a countable set. (3.3) follows from [**1**, Propositions 4.10 and 4.11].

LEMMA 2. *If* (3.1.1) *satisfies condition* A, *then* $F^{k+1}(x)$ *is* C_{k+1}*-continuous for* $x \in B$.

PROOF. By $(k-1)$ integrations by parts (cf. [**11**, p. 169]) we obtain

$$\begin{aligned}
&\frac{(k+1)!}{h^{k+1}}\left(\frac{1}{k!}\right)C_kP\int_x^{x+h}(x+h-t)^kF^{k+1}(t)\,dt \\
&\quad= \frac{(k+1)!}{h^{k+1}}\left(\frac{1}{k!}\right)C_kP\int_x^{x+h}(x+h-t)^kF_{(k+1)}(t)\,dt \\
&\quad= \frac{(k+1)!}{n^{k+1}}\left[F(x+h) - F(x) - \sum_{i=1}^{k}\frac{h^i}{i!}F_{(i)}(x)\right] \\
&\quad\to F_{(k+1)}(x) = F^{k+1}(x), \quad \text{for } x \in B,
\end{aligned}$$

by assuming, without loss of generality, that $F_{(r)}(0) = 0$, $0 \leqslant r \leqslant k$.

LEMMA 3. *If* (3.1.1) *satisfies condition* A *then* $f(x)$ *is* $SC_{k+1}P$*-integrable with base* B *on* $[\alpha, \alpha+2\pi]$, $\alpha \in B$, *and*

$$F^{k+1}(x)\Big|_\alpha^{\alpha+2\pi} = SC_{k+1}P\int_{[\alpha,\alpha+2\pi]}^{B} f(t)\,dt. \tag{3.4}$$

PROOF. We have by Lemma 2 that $F^{k+1}(x)$ is C_{k+1}-continuous in B. Since $F(x)$ is $(k+2)$-smooth everywhere in $(0, 2\pi)$ [**12**, Theorem 3.1] and $D_{k-2i}F(x) = F_{(k-2i)}(x)$, $0 \leqslant i \leqslant [k/2]$, it follows [**3**, Remark, p. 1279], that F is SC_{k+1}-continuous everywhere in $(0, 2\pi)$.

Moreover, since

$$\begin{aligned}
&C_{k+1}(F^{k+1}; x, x+h) - C_{k+1}(F^{k+1}; x, x-h) \\
&\qquad = \frac{2}{k+2}\,h\theta_{k+2}(F; x, h),
\end{aligned} \tag{3.5}$$

it is clear that

$$SC_{k+1}DF^{k+1}(x) = \lim \theta_{k+2}(F; x, h) = D_{k+2}F(x) = f(x), \qquad x \in B.$$

Since it is known [**12**, Theorem 5.1] that the set of points for which either

$$\underline{\lim}\ \theta_{k+2}(F; x, h) = -\infty \quad \text{or} \quad \overline{\lim}\ \theta_{k+2}(F; x, h) = +\infty$$

is a scattered set, it follows from (3.5) that the set of points for which either

$$SC_{k+1}D_*F^{k+1}(x) = -\infty \quad \text{or} \quad SC_{k+1}D^*F^{k+1}(x) = +\infty$$

is a scattered set.

We see then that $F^{k+1}(x) - F^{k+1}(\alpha)$ is both an $SC_{k+1}P$-major and an $SC_{k+1}P$-minor function of f on $[a, b]$ with base B and (3.4) holds.

If (3.1.1) satisfies condition A then its formal product with $\cos px$, $p = 1, 2, \ldots,$

$$\frac{1}{2}\, u_0 + \sum_{n=1}^{\infty} (u_n \cos nx + v_n \sin nx) \equiv \sum_{n=0}^{\infty} u_n(x), \tag{3.6}$$

and the formal product of (3.2) with $\cos px$,

$$\sum_{n=1}^{\infty} (+v_n \cos nx - u_n \sin nx) \equiv -\sum_{n=0}^{\infty} v_n(x) \tag{3.7}$$

are summable (C, k) in B, the former to sum $f(x) \cos px$ [**12**]. Consequently, the series obtained by integrating (3.6) formally term-by-term,

$$\frac{1}{2}\, u_0 x + \sum_{n=1}^{\infty} \frac{v_n(x)}{n}, \tag{3.8}$$

is summable $(C, k-1)$ in B. In the following we denote the $(C, k-1)$ sum of (3.8) in B by $G^{k+1}(x)$. Similarly the sum of the series obtained by forming the product of (3.6) with $\sin px$ and integrating term-by-term will be denoted by $H^{k+1}(x)$.

Now using the same methods as in Lemma 3 we obtain

LEMMA 4. *If* (3.1.1) *satisfies condition* A, *then for* $\alpha \in B$, $f(x)\cos px$ *and* $f(x)\sin px$, $p = 1, 2, \ldots,$ *are* $SC_{k+1}P$*-integrable on* $[\alpha, \alpha + 2\pi]$ *for* $p = 1, 2, \ldots$ *with base* B. *Moreover,*

$$G^{k+1}(x)\Big|_{\alpha}^{\alpha+2\pi} = SC_{k+1}P \int_{[\alpha,\alpha+2\pi]}^{B} f(t) \cos pt\, dt,$$

and

$$H^{k+1}(x)\Big|_{\alpha}^{\alpha+2\pi} = SC_{k+1}P \int_{[\alpha,\alpha+2\pi]}^{B} f(t) \sin pt\, dt.$$

THEOREM 1. *If* (3.1.1) *satisfies condition* A *then*

$$a_p = \frac{1}{\pi}\, SC_{k+1}P \int_{[\alpha,\alpha+2\pi]}^{B} f(t)\cos pt\, dt, \qquad p = 0, 1, 2, \ldots,$$

and

$$b_p = \frac{1}{\pi}\, SC_{k+1}P \int_{[\alpha,\alpha+2\pi]}^{B} f(t)\sin pt\, dt, \qquad p = 1, 2, \ldots.$$

PROOF. In the case of a_0 we have, by Lemma 3,

$$\frac{a_0 x}{2}\Big|_{\alpha}^{\alpha+2\pi} = a_0\pi = SC_{k+1}P\int_{[\alpha,\alpha+2\pi]}^{B} f(t)\cos nt\,dt.$$

The result for a_p follows in analogous fashion from Lemma 4 since $\frac{1}{2}u_0 = \frac{1}{2}a_p$, and the result for b_p is obtained in an exactly similar way.

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