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DOUBLE COMMUTANTS OF $C_{\cdot 0}$ CONTRACTIONS

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ABSTRACT. D. Sarason has shown that an operator in the double commutant of a contraction T of class $C_0(1)$ is interpolated by a function in H^∞, that is, $\{T\}' = \{\phi(T); \phi \in H^\infty\}$, [3]. Generally, in [4] Sz.-Nagy and C. Foiaş have shown that an operator in the double commutant of a contraction T of class $C_0(n)$ is interpolated by a function in N_T, that is, $\{T\}'' = \{\phi_1(T)^{-1}\phi_2(T): \phi_2/\phi_1 \in N_T\}$. In this note we shall show that an operator in the double commutant of a $C_{\cdot 0}$ contraction T with finite defect indices $\delta_{T^*} > \delta_T$ is interpolated by a function in H^∞.

1. Introduction. We begin by recalling notations of [6]. If T is a contraction on a separable Hilbert space H such that $T^{*n} \to 0$ (strongly) as $n \to \infty$, then T is said to be of class $C_{\cdot 0}$. If T and T^* are of class $C_{\cdot 0}$, then T is said to be of class C_{00}. For a contraction T, $\delta_T = \operatorname{rank}(1 - T^*T)$ and $\delta_{T^*} = \operatorname{rank}(1 - TT^*)$ are called *defect indices* of T. If T is of class C_{00} and $\delta_T = \delta_{T^*} = n < \infty$, then T is said to be of class $C_0(n)$. If T is of class $C_{\cdot 0}$ and $\delta_T = \delta_{T^*} = n < \infty$, then T is of class $C_0(n)$.

Let Θ be an $n \times m$ matrix over the Hardy space H^∞ of bounded measurable functions on the unit circle with vanishing Fourier coefficients of negative indices. Such a matrix is called *inner* if $\Theta^*(\lambda)\Theta(\lambda) = 1_m$ a.e. on the unit circle. In this case it necessarily follows that $n \geq m$. For T of class $C_{\cdot 0}$ with finite defect indices $\delta_{T^*} = n$ and $\delta_T = m$, there exists an $n \times m$ inner function Θ such that T is unitarily equivalent to $S(\Theta)$ on $H(\Theta)$, which are defined by next relations:

$$H(\Theta) = H_n^2 \ominus \Theta H_m^2$$

and

$$S(\Theta)h = P_\Theta SH \quad \text{for } h \text{ in } H(\Theta),$$

where H_n^2 is the Hardy space of n-dimensional column vector valued functions on the unit circle, P_Θ is the orthogonal projection of H_n^2 onto $H(\Theta)$, and $(Sh)(\lambda) = \lambda h(\lambda)$.

An $n \times m$ ($n \geq m$) *normal inner matrix* N' over H^∞ is, by definition, of the form:

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$$N' = \begin{bmatrix} \nu_1 & 0 \cdots\cdots & 0 \\ 0 & \nu_2 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots\cdots & \nu_m \\ 0 & \cdots\cdots & 0 \\ 0 & \cdots\cdots & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \Big\} n - m \end{matrix} \tag{1}$$

where for each i, ν_i is a scalar inner function and a divisor of its successor.

Let Θ be an $n \times m$ $(\infty > n \geqslant m)$ inner matrix over H^∞ and N a "*corresponding*" normal matrix i.e., N is an $n \times m$ normal inner matrix of the form (1), where each entry ν_i is the invariant factor of Θ, that is,

$$\nu_i = \frac{d_i}{d_{i-1}} \quad \text{for } i = 1, 2, \ldots, m,$$

where $d_0 = 1$ and d_i is the largest common inner divisor of the minors of order i. Then Nordgren [2] has shown that there exist $n \times n$ matrices Δ, Δ_k and $m \times m$ matrices Λ, Λ_k $(k = 1, 2)$ satisfying

$$\Delta\Theta = N\Lambda, \qquad \Delta_k N = \Theta\Lambda_k,$$
$$(\det \Delta)(\det \Lambda) \wedge d_m = 1, \qquad (\det \Delta_k)(\det \Lambda_k) \wedge d_m = 1,$$
$$(\det \Delta_1) \wedge (\det \Delta_2) = 1, \tag{2}$$

where $x \wedge y$ denotes the largest common inner divisor of the scalar functions x and y in H^∞. Setting $X = P_N\Delta|H(\Theta)$ and $Y_k = P_\Theta\Delta_k|H(N)$, one obtains one-to-one operators X, Y_1 and Y_2 such that

$$XS(\Theta) = S(N)X, \qquad Y_kS(N) = S(\Theta)Y_k \quad \text{and} \tag{3}$$
$$Y_1H(N) \vee Y_2H(N) = H(\Theta), \tag{4}$$

where $L_1 \vee L_2$ denotes the minimum subspace including both L_1 and L_2 (see [5]).

For a completely nonunitary contraction T, it is possible to define $\phi(T)$ for every function ϕ in H^∞. In particular, for $S(\Theta)$ given above, $\phi(S(\Theta))$ can be equivalently defined by the following:

$\phi(S(\Theta))h = P_\Theta\phi h$ for h in $H(\Theta)$ (see [3], [6]).

The purpose of this note is to prove

THEOREM. *Let Θ be an $n \times m$ inner matrix over H^∞ $(n > m)$. Then*

$$\{S(\Theta)\}'' = \{\phi(S(\Theta)) : \phi \in H^\infty\}.$$

2. Proof of the theorem. The following lemma is well known and called the *lifting theorem.*

LEMMA 1 ([3], [6]). *Let Θ be an $n \times m$ inner matrix over H^∞. Then for an operator T commuting with $S(\Theta)$ there is an $n \times n$ matrix Ψ over H^∞ such that*

$$\Psi\Theta H_m^2 \subseteq \Theta H_m^2 \quad \textit{and} \quad T = P_\Theta \Psi | H(\Theta).$$

The following lemma is analogous to Lemma 1 of [7] and the proof is simple. Thus we omit it.

LEMMA 2. *Let N' be an $n \times m$ normal inner matrix of the form* (1). *In order that an $n \times n$ matrix $\Psi = (\psi_{ij})$ over H^∞ satisfies $\Psi N' H_m^2 \subseteq N' H_m^2$, it is necessary and sufficient that*

(i) *ν_i / ν_j is a divisor of ψ_{ij} for $1 \leqslant j < i \leqslant m$,*

(ii) *$\psi_{ij} = 0$ for $m + 1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$.*

LEMMA 3. *If T belongs to $\{S(\Theta)\}''$, then there exist ϕ_1 and ϕ_2 in H^∞ such that*

$$\phi_2(S(\Theta)) = \phi_1(S(\Theta))T \quad \textit{and} \quad \phi_1 \wedge d_m = 1. \tag{5}$$

PROOF. Let N be an $n \times m$ normal inner matrix of the form (1) corresponding to Θ. Let us consider X, Y_1 and Y_2 defined above. Then we have

$$Y_k A X \in \{S(\Theta)\}' \quad \text{for every } A \in \{S(N)\}',$$

and hence $(Y_k A X)T = T(Y_k A X)$ $(k = 1, 2)$. Thus, on setting $B_k = XY_k$ and $C_k = XTY_k$, one obtains

$$B_k A C_k = C_k A B_k \qquad (k = 1, 2). \tag{6}$$

By Lemma 1, $T \in \{S(\Theta)\}'$ and $A \in \{S(N)\}'$ imply that there are $n \times n$ matrices Γ and Ψ such that

$$\Gamma\Theta H_m^2 \subseteq \Theta H_m^2, \qquad T = P_\Theta \Gamma | H(\Theta),$$
$$\Psi N H_m^2 \subseteq N H_m^2 \quad \text{and} \quad A = P_N \Psi | H(N).$$

Then it is obvious that $B_k = P_N \Delta\Delta_k' | H(N)$ and $C_k = P_N \Delta\Gamma\Delta_k' | H(N)$. Now let ψ_{ij}, b_{ij}^k and c_{ij}^k be the (i, j)th entry of Ψ, $\Delta\Delta_k$ and $\Delta\Gamma\Delta_k$, respectively. Since (6) implies that

$$\{(\Delta\Delta_k)\Psi(\Delta\Gamma\Delta_k) - (\Delta\Gamma\Delta_k)\Psi(\Delta\Delta_k)\} H_n^2 \subseteq N H_m^2,$$

it follows that for $1 \leqslant i, j \leqslant n$

$$\sum_{h,l=1}^{n} \left\{ b_{ih}^k \psi_{hl} c_{lj}^k - c_{ih}^k \psi_{hl} b_{lj}^k \right\} \in \nu_i H^\infty, \tag{7}$$

where $\nu_{m+1} \equiv \cdots \equiv \nu_n \equiv 0$.

Set $\psi_{ij} = 1$ for $(i, j) = (1, n)$ and $\psi_{ij} = 0$ for $(i, j) \neq (1, n)$. Then, since ψ_{ij} $(1 \leqslant i, j \leqslant n)$ satisfy the conditions (i) and (ii), by (7) we have

$$b_{i1}^k c_{nj}^k - c_{i1}^k b_{nj}^k \in \nu_i H^\infty \quad \text{for } 1 \leqslant i, j \leqslant n.$$

Similarly we can deduce that

$$b_{ir}^k c_{nj}^k - c_{ir}^k b_{nj}^k \in \nu_i H^\infty \quad \text{for } 1 \leqslant i, j, r \leqslant n. \tag{8}$$

Thus it is clear that $(c_{nj}^k \Delta\Delta_k - b_{nj}^k \Delta\Gamma\Delta_k) H_n^2 \subseteq N H_m^2$, which implies that

$$c_{nj}^k(S(N))B_k - b_{nj}^k(S(N))C_k = 0. \tag{9}$$

From (3) and (9) it follows that

$$Xc_{nj}^k(S(\Theta))Y_k = XY_kc_{nj}^k(S(N)) = B_kc_{nj}^k(S(N))$$
$$= b_{nj}^k(S(N))C_k = b_{nj}^k(S(N))XTY_k = Xb_{nj}^k(S(\Theta))TY_k.$$

Since X is one-to-one, we have

$$c_{nj}^k(S(\Theta))Y_k = b_{nj}^k(S(\Theta))TY_k. \tag{10}$$

(Remark. The above method is analogous to that in **[1]** or **[4]**.) Let $\tilde{b}_{ij}^k$ be the (i, j)th cofactor of $\Delta\Delta_k$. Then from (10) it follows that

$$\left(\sum_{j=1}^{n} \tilde{b}_{nj}^k c_{nj}^k\right)(S(\Theta))Y_k = (\det \Delta\Delta_k)(S(\Theta))TY_k. \tag{11}$$

Let δ_{ij}, δ_{ij}^k and γ_{ij} be the (i, j)th entries of Δ, Δ_k and Γ, respectively. Then since

$$b_{ij}^k = \sum_{l=1}^{n} \delta_{il}\delta_{lj}^k \quad \text{and} \quad c_{nj}^k = \sum_{h,l=1}^{n} \delta_{nh}\gamma_{hl}\delta_{lj}^k,$$

we have

$$\begin{bmatrix} b_{11}^k & ,\dots, & b_{1n}^k \\ \vdots & & \vdots \\ b_{n-11}^k & ,\dots, & b_{n-1n}^k \\ c_{n1}^k & ,\dots, & c_{nn}^k \end{bmatrix} = \Pi\Delta_k, \quad \text{where } \Pi = \begin{bmatrix} \delta_{11} & ,\dots, & \delta_{1n} \\ \vdots & & \vdots \\ \delta_{n-11} & ,\dots, & \delta_{n-1n} \\ \sum_h \delta_{nh}\gamma_{h1} & ,\dots, & \sum_h \delta_{nh}\gamma_{hn} \end{bmatrix}.$$

Thus from (11) it follows that

$$(\det \Delta_k)(S(\Theta)) \cdot (\det \Pi)(S(\Theta))Y_k = (\det \Delta_k)(S(\Theta)) \cdot (\det \Delta)(S(\Theta))TY_k.$$

The second equation of (2) implies that $(\det \Delta_k)(S(\Theta))$ is one-to-one (see **[8]**). Hence it follows that

$$(\det \Pi)(S(\Theta))Y_k = (\det \Delta)(S(\Theta))TY_k.$$

From this and (4) it is clear that

$$(\det \Pi)(S(\Theta)) = (\det \Delta)(S(\Theta))T.$$

Consequently, if $\phi_1 = \det \Delta$ and $\phi_2 = \det \Pi$, then this and the first equation of (2) imply that Lemma 3 is true.

PROOF OF THEOREM. $\phi_2(S(\Theta)) = \phi_1(S(\Theta))T$ implies that there exists an $m \times n$ matrix $\Omega = (\omega_{ij})$ over H^∞ such that $\phi_2 - \phi_1\Gamma = \Theta\Omega$. Setting θ_{ij} the (i, j)th entry of Θ, we have

$$-\phi_1\gamma_{ij} = \sum_{k=1}^{m} \theta_{ik}\omega_{kj} \quad \text{for } i \neq j \tag{12}$$

and

$$\phi_2 - \phi_1\gamma_{ii} = \sum_{k=1}^{m} \theta_{ik}\omega_{ki} \quad \text{for } i = 1, 2, \dots, n. \tag{13}$$

From (5), there is no loss of generality in assuming $\phi_1 \wedge \phi_2 = 1$. Then for every minor ξ_α of Θ with order m, it follows that the inner factor of ϕ_1 is a divisor of ξ_α. In fact, if $\xi_\alpha \equiv 0$, then it is obvious. Thus assume $\xi_\alpha \not\equiv 0$ and $\xi_\alpha = \det \Theta_\alpha$, where

$$\Theta_\alpha = \begin{bmatrix} \theta_{\alpha(1)1} & ,\dots, & \theta_{\alpha(1)m} \\ \vdots & & \vdots \\ \theta_{\alpha(m)1} & ,\dots, & \theta_{\alpha(m)m} \end{bmatrix} \quad \text{for } 1 \leqslant \alpha(1) < \cdots < \alpha(m) \leqslant n.$$

Then $n > m$ implies that there is an l such that $1 \leqslant l \leqslant n$ and $l \neq \alpha(k)$ for $k = 1, 2, \dots, m$. From (12) it follows that

$$-\phi_1 \begin{bmatrix} \gamma_{\alpha(1)l} \\ \vdots \\ \gamma_{\alpha(m)l} \end{bmatrix} = \Theta_\alpha \begin{bmatrix} \omega_{1l} \\ \vdots \\ \omega_{ml} \end{bmatrix}.$$

Now, since $\det \Theta_\alpha = \xi_\alpha \not\equiv 0$, there exists an $m \times m$ matrix Θ'_α over H^∞ such that $\Theta_\alpha \Theta'_\alpha = \Theta'_\alpha \Theta_\alpha = \det \Theta_\alpha = \xi_\alpha$. Thus it is clear that

$$-\phi_1 \Theta'_\alpha \begin{bmatrix} \gamma_{\alpha(1)l} \\ \vdots \\ \gamma_{\alpha(m)l} \end{bmatrix} = \xi_\alpha \begin{bmatrix} \omega_{1l} \\ \vdots \\ \omega_{ml} \end{bmatrix},$$

which implies that ϕ_1 is a divisor of $\xi_\alpha \omega_{il}$ for $i = 1, 2, \dots, m$. Hence, by (13), ϕ_1 is a divisor of

$$\xi_\alpha (\theta_{l1}\omega_{1l} + \theta_{l2}\omega_{2l} + \cdots + \theta_{lm}\omega_{ml}) = \xi_\alpha (\phi_2 - \phi_1 \gamma_{ll}).$$

From $\phi_1 \wedge \phi_2 = 1$, this implies that the inner factor of ϕ_1 is a divisor of ξ_α. Thus it is a divisor of $\bigwedge_\alpha \xi_\alpha = d_m$; this, from $\phi_1 \wedge d_m = 1$, yields that the inner factor of $\phi_1 = 1$ i.e., ϕ_1 is outer. Since ϕ_1 is a divisor of $\xi_\alpha \phi_2$, there exist ζ_α in H^∞ such that $\phi_1 \zeta_\alpha = \xi_\alpha \phi_2$. Since Θ is inner $\Sigma_\alpha |\xi_\alpha|^2 = 1$ a.e. on the unit circle [5]. Thus,

$$|\phi_2| = |\phi_1| \left(\sum |\zeta_\alpha|^2 \right)^{1/2} \leqslant |\phi_1| \left(\sum \|\zeta_\alpha\|_\infty \right) \text{ a.e.}$$

implies that there exists a ϕ in H^∞ such that $\phi_1 \phi = \phi_2$.

Thus we have $\phi_1(\phi - \Gamma) = \Theta\Omega$, which implies that

$$\phi_1(S(\Theta))\{\phi(S(\Theta)) - T\} = 0$$

and hence $T = \phi(S(\Theta))$. Thus we complete the proof of Theorem.

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