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# ON THE NOTION OF $n$-CARDINALITY

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ABSTRACT. In this paper we introduce and investigate the notion of *n-cardinality*, which turned out to be useful in constructions involving product spaces and has a number of interesting applications.

In this paper we introduce and investigate the notion of *n-cardinality*, which turned out to be useful in constructions involving product spaces and has a number of interesting applications (see **[P]**, **[P$_1$]**, **[P$_2$]**, and **[P$_3$]**).

The notion of $n$-cardinality arose after discussions with Eric van Douwen, who also proved an important case of Theorem 1 (Corollary 1). The author is grateful to him for his valuable suggestions.

Throughout this paper $n$ denotes a natural number and $c = 2^\omega$. Let $X$ be an arbitrary set. For a point $p = (p_1, \ldots, p_n)$ from $X^n$ by $\hat{p}$ we shall denote the *set* $\{p_1, \ldots, p_n\}$ of coordinates of $p$. By $p_i$ we shall always mean the $i$th coordinate of $p$. For undefined notions and symbols the reader is referred to **[E]**.

LEMMA 1. *For a subset $A$ of $X^n$ the following cardinals are well defined and they are equal provided that one of them or–equivalently–all of them are infinite*:

(i) $\max\{|B|\colon B \subset A$ *and* $p_i \neq q_i$, *for* $i = 1, 2, \ldots, n$ *and every two distinct points $p$ and $q$ from* $B\}$;

(ii) $\max\{|B|\colon B \subset A$ *and* $\hat{p} \cap \hat{q} = \varnothing$, *for every two distinct points $p$ and $q$ from* $B\}$;

(iii) $\min\{|Y|\colon Y \subset X$ *and* $A \subset \bigcup_{i=1}^{n}(X^{i-1} \times Y \times X^{n-i})\}$.

PROOF. Let us denote by $\tau$ the cardinal number defined in (iii). Since $\tau$ is well defined it suffices to show that: (a) if $\tau \geqslant \omega$, then cardinals described in (i) and (ii) coincide with $\tau$ and (b) if $\tau < \omega$, then cardinals described in (i) and (ii) are finite.

Let us note first that if $B$ is a subset of $A$ such that $p_i \neq q_i$, for $i = 1, 2, \ldots, n$ and every two distinct points $p$ and $q$ from $B$, then $|B| \leqslant n \cdot \tau$. Indeed, let $Y$ be a subset of $X$ of cardinality $\tau$ such that $A \subset \bigcup_{i=1}^{n}(X^{i-1} \times Y \times X^{n-i})$. For every $i = 1, 2, \ldots, n$ and every $y \in Y$ there exists at most one $p \in B$ such that $p_i = y$, therefore $|B| \leqslant n \cdot \tau$. From this fact we deduce (b) and infer that in order to prove (a) it suffices to construct a subset $B$ of $A$

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of cardinality $\tau$ such that $\hat{p} \cap \hat{q} = \varnothing$, for every two distinct points $p, q \in B$.

Assume that $\tau \geqslant \omega$. We shall construct points $p(\alpha)$ of $B$, for $\alpha < \tau$, by transfinite recursion. Assume that points $p(\beta) \in A$ have been constructed for $\beta < \alpha$ so that $\hat{p}(\beta) \cap \hat{p}(\gamma) = \varnothing$, if $\beta \neq \gamma$.

The set $Z = \bigcup \{\hat{p}(\beta)\colon \beta < \alpha\}$ has cardinality $< \tau$ and therefore there exists a point

$$p(\alpha) \in A \setminus \bigcup_{i=1}^{n} (X^{i-1} \times Z \times X^{n-i}).$$

Clearly $\hat{p}(\alpha) \cap \hat{p}(\beta) = \varnothing$, for every $\beta < \alpha$, which completes the proof of the Lemma. □

Definition 1. For a subset $A$ of $X^n$, where $X$ is an arbitrary set, we define the *n-cardinality* $|A|_n$ of $A$ (with respect to $X^n$) by $|A|_n = \max\{|B|\colon B \subset A$ and $p_i \neq q_i$, for every two distinct points $p$ and $q$ from $B$ and $i = 1, 2, \ldots, n\}$. We say that $A$ is *n-countable* (*n-uncountable*) if $|A|_n \leqslant \omega$ ($|A|_n > \omega$). □

It follows from Lemma 1 that $n$-cardinality is well defined and moreover:

(1) $|A|_1 = |A|$; i.e. $n$-cardinality generalizes the notion of cardinality;

(2) $$|A|_n \leqslant |A|;$$

(3) $$|A|_2 = \min\{|Y|\colon A \subset Y \times X \cup X \times Y\},$$ provided that $|A|_2$ is infinite.

Remark 1. We can analogously define the $n$-cardinality of a subset $A$ of $\prod_{i=1}^n X_i$, where $X_i$'s are arbitrary sets, however, this potentially more general definition can be reduced to the previous one by observing that the so defined $n$-cardinality coincides with the $n$-cardinality of $A$ with respect to $X^n$, where $X = \bigoplus_{i=1}^n X_i$. Making use of this observation, one can easily show that all results proved in this paper for subsets of $X^n$ are actually valid–after obvious modifications–for subsets of the products $\prod_{i=1}^n X_i$. □

The following theorem generalizes a result of van Douwen (see Corollary 1).

Theorem 1 (Main). *Let $X$ be a complete separable metric space and $B$ a Borel subset of $X^n$. The following statements are equivalent*:

(i) *$B$ is n-uncountable*;

(ii) *$B$ has n-cardinality continuum*;

(iii) *$B$ contains a homeomorphic image $h(C)$ of the Cantor set $C$ such that $\hat{h}(x) \cap \hat{h}(y) = \varnothing$, for $x \neq y$*;

(iv) *$B$ contains a homeomorphic image $h(C)$ of the Cantor set $C$ by the diagonal*

$$h = \mathop{\triangle}_{i=1}^{n} h_i\colon C \to X^n$$

*of homeomorphic embeddings $h_i\colon C \to X$*;

(v) (*for $n > 1$*) *$B$ contains the graph of a homeomorphic embedding $h$*:

$C \to X^{n-1}$ *of a Cantor subset* $C$ *of* $X$ *into* $X^{n-1}$ *such that* $\hat{h}(x) \cap \hat{h}(y) = \varnothing$, *for* $x \neq y$;

(vi) (*for* $n > 1$) $B$ *contains the graph of the diagonal* $h = \triangle_{i=2}^{n} h_i\colon C \to X^{n-1}$ *of homeomorphic embeddings* $h_i\colon C \to X$ *of a Cantor subset* $C$ *of* $X$ *into* $X$.

PROOF. It follows immediately from Lemma 1 that either of the conditions (ii)–(vi) implies (i). We shall show the converse.

It is known that every Borel subset of a separable complete metric space is a continuous image of the space $P$ of irrationals (cf. [**K**, Theorem 1, Chapter III, §37]). Let $f\colon P \to B$ be a continuous mapping of $P$ onto $B$ and assume that $|B|_n > \omega$. Let us choose an arbitrary complete metric on $P$. By Lemma 1 there exists a collection $\{p(s)\}_{s \in S}$ of points $B$ such that $\hat{p}(s) \cap \hat{p}(s') = \varnothing$, for $s \neq s'$ and $|S| = \omega_1$.

For each $s \in S$ choose an $x_s \in f^{-1}(p(s))$ and put $T = \{x_s\}_{s \in S}$. Without loss of generality we can assume that $T$ is dense-in-itself (otherwise, since $T$ is second countable, by the Bernstein Theorem we would remove countably many points from $S$ and $T$).

For each $m = 1, 2, \ldots$ and every sequence $(d_1, \ldots, d_m)$, where $d_i = 0$ or 1, we will define a point $t(d_1, \ldots, d_m) \in T$ and a closed ball $B(d_1, \ldots, d_m)$ in $P$ with the center at the point $t(d_1, \ldots, d_m)$ and radius $< 1/m$ so that:

$$(4)_m \qquad B(d_1, \ldots, d_m) \subset B(d_1, \ldots, d_{m-1}), \quad \text{for } m > 1;$$

$(5)_m$ for each pair $(d_1, \ldots, d_m)$ and $(d_1', \ldots, d_m')$ of distinct sequences there exist disjoint subsets $V_0$ and $V_1$ of $X$ such that $f(B(d_1, \ldots, d_m)) \subset V_0^n$ and $f(B(d_1', \ldots, d_m')) \subset V_1^n$.

Let $m = 1$ and choose two distinct points $t(0)$ and $t(1)$ from $T$. Since $X$ is Hausdorff, there exist disjoint open subsets $V_0$ and $V_1$ of $X$ with $\hat{f}(t(j)) \subset V_j$, for $j = 0, 1$. By the continuity of $f$ there exist closed balls $B(0)$ and $B(1)$ with centers at $t(0)$ and $t(1)$, respectively, and radii $< 1$ such that $f(B(j)) \subset V_j^n$, for $j = 0, 1$, which completes the first step of the inductive construction.

Assume that $m \geqslant 2$ and that an inductive step has been made for $m - 1$. Let us take an arbitrary sequence $(d_1, \ldots, d_{m-1})$ and find two distinct points $t_j = t(d_1, \ldots, d_{m-1}, j)$, $j = 0, 1$, from $T$ belonging to the interior of $B(d_1, \ldots, d_{m-1})$. Such points exist because $T$ is dense-in-itself. We can find two disjoint open subsets $V_j, j = 0, 1$, of $X$ such that

$$\hat{f}\,(t(d_1, \ldots, d_{m-1}, j)) \subset V_j, \qquad j = 0, 1.$$

There exist closed balls $B(d_1, \ldots, d_{m-1}, j)$, $j = 0, 1$, with centers at the points $t_j$ and radii $< 1/m$ such that

$$B(d_1, \ldots, d_{m-1}, j) \subset B(d_1, \ldots, d_{m-1})$$

and

$$f(B(d_1, \ldots, d_{m-1}, j)) \subset V_j^n, \qquad j = 0, 1.$$

It is easy to see that the conditions $(4)_m$ and $(5)_m$ are satisfied, which completes our inductive construction.

One easily sees that the subset

$$C = \bigcap_{m=1}^{\infty} \left( \bigcup \{B(d_1, \ldots, d_m): (d_1, \ldots, d_m) \in \{0, 1\}^m\}\right)$$

of $P$ is homeomorphic to the Cantor set (cf. [**K**, Chapter III, §36,I]) and that the continuous mapping $h = f|C: C \to B \subset X^n$ has the property

$$\hat{h}(x) \cap \hat{h}(y) = \varnothing, \quad \text{for } x \neq y,$$

in particular, $h$ is one-to-one. As a one-to-one continuous mapping into a Hausdorff space defined on a compact space $C$, the mapping $h$ is a homeomorphic embedding. Therefore, (iii) is satisfied and consequently, by Lemma 1, also (ii) follows.

Let $h(x) = (h_1(x), \ldots, h_n(x))$, for $x \in C$. Since the mappings $h_i$ are continuous and one-to-one, they are homeomorphic embeddings and (iv) holds.

Assume that $n > 1$ and let $C^1 = h_1(C) \subset X$ and $h_i^1 = h_i \circ h_1^{-1}: C^1 \to X$, for $i = 2, 3, \ldots, n$. Clearly $C^1$ is homeomorphic to the Cantor set, $h_i^1$'s are homeomorphic embeddings of $C^1 \subset X$ into $X$ and the graph of the diagonal $h^1 = \triangle_{i=2}^n h_i^1: C^1 \to X^{n-1}$ coincides with $h(C)$. This shows that also conditions (v) and (vi) are satisfied and completes the proof. □

REMARK 2. It follows from the above proof that conditions (i)–(vi) are actually equivalent for every *analytic* subset $B$ of $X^n$, where $X$ is an arbitrary Hausdorff space (analytic sets are continuous images of irrationals). □

REMARK 3. R. Pol pointed out that Theorem 1 (and also Theorem 3) can be derived from the results obtained recently by K. Kuratowski [$\mathbf{K_1}$, Corollary 3], however, the direct proof of these theorems seems to be simpler. □

The following corollary has been first proved by van Douwen [**vD**].

COROLLARY 1. *Let $X$ be a separable complete metric space. A closed subset $F$ of $X^n$ is either $n$-countable or has $n$-cardinality continuum.* □

COROLLARY 2. *Let $X$ be a separable complete metric space. A Borel subset $B$ of $X^2$ is either contained in $(X \times A) \cup (A \times X)$, with $A$ countable, or it contains a graph of a homeomorphic embedding $h: C \to X$ of a Cantor subset $C$ of $X$ into $X$.* □

COROLLARY 3 (ALEXANDROV-HAUSDORFF). *Every uncountable Borel subset of a separable complete metric space contains a Cantor set $C$ and therefore, has cardinality continuum.* □

The next theorem (and its corollary) generalizes the classical theorem of Bernstein (cf. [**K**, Theorem 1, §40, I]) on the existence of totally imperfect subsets of the real line and plays an important role in applications of $n$-cardinality (see [**P**], [$\mathbf{P_1}$], [$\mathbf{P_2}$], and [$\mathbf{P_3}$]).

THEOREM 2. *Let $X$ be a separable complete metric space. There exist disjoint* *subsets $A_i$ of $X$, where $i < \omega$, such that for every $n < \omega$, every $n$-uncountable Borel subset $B$ of $X^n$ and every $i < \omega$ we have*

$$|B \cap A_i^n|_n = 2^\omega.$$

PROOF. Let us denote by $\mathfrak{B}_n$ the family of all $n$-uncountable Borel subsets of $X^n$. Since there are at most continuum Borel subsets in a separable metric space, the cardinality of $\mathfrak{B}_n$ is $\leqslant c$. Let $\{B_\alpha\}_{\alpha<c}$ be such an enumeration of all elements of $\mathfrak{B} = \bigcup_{n=1}^{\infty}\mathfrak{B}_n$ that every element from $\mathfrak{B}$ is listed continuum many times. For each $\alpha < c$ there exists exactly one $n(\alpha)$ such that $B_\alpha \in \mathfrak{B}_{n(\alpha)}$.

For $\alpha < c$ and $i < \omega$ we will construct points $p(\alpha, i)$ belonging to $B_\alpha$ in such a way that

$$\hat{p}(\alpha, i) \cap \hat{p}(\alpha', i') = \varnothing, \quad \text{if } (\alpha, i) \neq (\alpha', i'). \tag{6}$$

Let $p(0, i)$, $i < \omega$, be arbitrary points from $B_0$ such that $\hat{p}(0, i) \cap \hat{p}(0, i') = \varnothing$, if $i \neq i'$. Such points exist because $B_0$ is $n(0)$-uncountable. Let us take $\alpha < c$ and assume that we have already constructed points $p(\beta, i)$, for $\beta < \alpha$ and $i < \omega$. The set $Y = \bigcup \{\hat{p}(\beta, i)\colon \beta < \alpha, i < \omega\}$ has cardinality less than $c$ and therefore by Theorem 1 the set

$$B_\alpha^* = B_\alpha \setminus \bigcup_{j=1}^{n} (X^{j-1} \times Y \times X^{n-j}),$$

where $n = n(\alpha)$, has $n$-cardinality continuum and consequently we can find for $i < \omega$ points $p(\alpha, i) \in B_\alpha^*$, such that $\hat{p}(\alpha, i) \cap \hat{p}(\alpha, i') = \varnothing$, if $i \neq i'$ which completes the inductive construction. It is easy to see that (6) is satisfied.

Let us put $A_i = \bigcup_{\alpha<c}\{\hat{p}(\alpha, i)\}$. Clearly the sets $A_i$, $i < \omega$, are disjoint. If $n < \omega$ and $B$ is an $n$-uncountable Borel subset of $X^n$ then there exist continuum many ordinals $\alpha < c$ such that $B = B_\alpha$ and for every such $\alpha$ and every $i < \omega$ we have

$$p(\alpha, i) \in B_\alpha \cap (\hat{p}(\alpha, i))^n \subset B \cap A_i^n.$$

It follows from Lemma 1 and (6) that $|B \cap A_i^n|_n = 2^\omega$. □

COROLLARY 4. *Let $X$ be a separable complete metric space. There exists a subset $A$ of $X$ such that for every $n < \omega$, the complement of any Borel subset of $X^n$ containing either $A^n$ or $(X \setminus A)^n$ is $n$-countable.*

PROOF. Let $A_i$'s be as in Theorem 2. Put $A = A_0$ and recall that the complement of a Borel set is a Borel set. □

The following theorem can be proved in a similar way as Theorem 1 using the Theorem of Arhangel'skiĭ [A].

THEOREM 3. *Let $X$ be a first countable complete Lindelöf space. A closed subset $F$ of $X^n$ is either $n$-countable or has $n$-cardinality continuum.* □

COROLLARY 5. *Let $X$ be a first countable compact space. A closed subset $F$ of $X^n$ is either $n$-countable or has $n$-cardinality continuum.* □

COROLLARY 6 (ČECH-POSPIŠIL-ARHANGEL'SKIĬ). *A first countable compact space is either countable or has cardinality continuum.* □

REMARK 4. Theorem 3 can be generalized in the following way: Let $X$ be a first countable Hausdorff space. A complete Lindelöf subspace $A$ of $X^n$ is either $n$-countable or has $n$-cardinality continuum. □

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