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AN IMPROVEMENT THEOREM FOR DESCARTES SYSTEMS

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ABSTRACT. An improvement (or comparison) theorem is proved for certain linear combinations of functions from a Descartes system. This theorem can then be applied to prove a conjecture of Lorentz, as well as more general results.

1. Introduction. The results in this paper were motivated by a problem posed by G. G. Lorentz. Lorentz was interested in minimizing $\|x^N - \sum_{i=1}^{k} a_i x^{\lambda_i}\|$ where the λ_i are integers, $0 \leqslant \lambda_i < N$, the a_i are real numbers, $k < N$, and $\|\cdot\|$ is the supremum norm on $[0, 1]$. It was conjectured by Lorentz that for given integers k and N the set of exponents λ_i which produced the smallest error is $\lambda_i = N - i$, $i = 1, \ldots, k$. This was proved in [1] by noting that the kernel $K(x, y) = x^y$ is extended totally positive (ETP) on $(0, \infty) \times (-\infty, \infty)$.

Subsequently, A. Pinkus [5] observed that this result is valid when $\|\cdot\|$ is any L_p norm, $1 \leqslant p \leqslant \infty$. His proof relied on the fact that $\{x^i\}_{i=0}^{N}$ is a Descartes system on $(0, 1)$.

These results are very striking since one might not expect the same set of exponents to produce the smallest error in all the L_p norms. The purpose of this paper is to expose the basic property of Descartes systems from which these and more general results flow, namely

THEOREM 1. *Let $\{u_i\}_{i=1}^{N} \subset C(c, d)$ be a Descartes system on (c, d). Let an integer $k \leqslant N - 2$ be given along with integers $N > \lambda_i \geqslant \gamma_i \geqslant 1$ for $i = 1, \ldots, k$. Suppose that $c < x_1 < \cdots < x_k < d$,*

$$p = u_N + \sum_{i=1}^{k} a_i u_{\lambda_i}, \qquad q = u_N + \sum_{i=1}^{k} b_i u_{\gamma_i},$$

and

$$0 = p(x_i) = q(x_i), \qquad i = 1, \ldots, k.$$

Then $|p(x)| \leqslant |q(x)|$ for all $x \in (c, d)$, with strict inequality if $x \neq x_i$, $i = 1, \ldots, k$, provided that $p \neq q$.

This theorem is reminiscent of the "improvement" theorems of Karlin [3].

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§2 contains the relevant definitions, a preliminary lemma, and the proof of Theorem 1. In addition, we state an extension of Theorem 1. §3 relates Theorem 1 and its extension to certain approximation results.

2. Proof of Theorem 1 and extensions. We begin with some necessary definitions and notations. A set of functions $\{u_i\}_{i=1}^N \subset C(c, d)$ will be called a Descartes system on (c, d) [2, pp. 25–27] provided there is an $\epsilon_k = \pm 1$ so that

$$0 < \epsilon_k \det \begin{bmatrix} u_{\lambda_1}(t_1) & u_{\lambda_2}(t_1) & \cdots & u_{\lambda_k}(t_1) \\ \vdots & & & \\ u_{\lambda_1}(t_k) & u_{\lambda_2}(t_k) & \cdots & u_{\lambda_k}(t_k) \end{bmatrix} \tag{2.1}$$

whenever $1 \leqslant \lambda_1 < \cdots < \lambda_k \leqslant N$, $c < t_1 < \cdots < t_k < d$, and $1 \leqslant k \leqslant N$.

The next lemma can be easily verified via Cramer's rule.

LEMMA. *Let $\{u_i\}_{i=1}^N$ be a Descartes system on (c, d), $2 \leqslant m \leqslant N$, integers $1 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_m \leqslant N$, and $c < x_1 < \cdots < x_{m-1} < d$ be given. Suppose that $p = \sum_{i=1}^m a_i u_{\lambda_i}$ is not zero but $p(x_i) = 0$ for $i = 1, \ldots, m-1$. Then*

(i) *$p(x) = 0$ only if $x = x_i$, $1 \leqslant i \leqslant m-1$.*

(ii) *p changes sign at each x_i.*

(iii) *$a_i a_{i+1} < 0$ for $i = 1, \ldots, m-1$.*

(iv) *$a_m p(x)\epsilon_m \epsilon_{m-1} > 0$ for $x_{m-1} < x < d$.*

This lemma is the key to proving Theorem 1. It provides the necessary information concerning the "orientation" of elements in the span of a Descartes system which vanish maximally.

We now proceed to prove Theorem 1. It is clear that we need only consider p and q of the form

$$p = u_N + \sum_{\substack{i=1 \\ i \neq j}}^{k} a_i u_{\lambda_i} + a u_\lambda, \qquad q = u_N + \sum_{\substack{i=1 \\ i \neq j}}^{k} b_i u_{\lambda_i} + b u_\gamma$$

where $1 \leqslant \lambda_1 < \cdots < \lambda_{j-1} < \gamma < \lambda < \lambda_{j+1} < \cdots < \lambda_k < N$, since the general result may be inferred from this case by making a finite number of pairwise comparisons.

With p and q as above the proof proceeds by showing that p and q have the same "orientation" (i.e. sign structure) but that $p - q$ has opposite "orientation". This will complete the proof. By hypothesis p and q have zeros at $x_1, \ldots, x_k$ and hence part (iv) of the Lemma implies that

$$p(x)\epsilon_{k+1}\epsilon_k > 0 \quad \text{and} \quad q(x)\epsilon_{k+1}e_k > 0 \tag{2.2}$$

for $x_k < x < d$. Furthermore,

$$p - q = \sum_{\substack{i=1 \\ i \neq j}}^{k} c_i u_{\lambda_i} + a u_\lambda - b u_\gamma$$

has $k+1$ terms and also vanishes maximally at $x_1, \ldots, x_k$. Thus, the coefficient of the leading term for $p-q$ (i.e. c_k if $j \neq k$) is negative by part (iii) of the Lemma (since p and $p-q$ have the same coefficient for u_λ). By part (iv) of the Lemma we have

$$(p-q)(x)\epsilon_{k+1}\epsilon_k < 0, \qquad x_k < x < d. \tag{2.3}$$

Combining (2.2) and (2.3) we have, for $x_k < x < d$

$$0 < p(x) < q(x) \quad \text{if } \epsilon_{k+1}\epsilon_k > 0$$

or

$$0 > p(x) > q(x) \quad \text{if } \epsilon_{k+1}\epsilon_k < 0.$$

In any case we see that $|p(x)| < |q(x)|$ for $x_k < x < d$.

Using part (ii) of the Lemma we note that p, q, and $p-q$ all change sign at x_k and hence for $x_{k-1} < x < x_k$ we have $|p(x)| < |q(x)|$. Applying this argument repeatedly completes the proof of Theorem 1.

Theorem 1 may be generalized as follows:

THEOREM 2. *Let* $\{u_i\}_{i=1}^N \subset C(a, b)$ *be a Descartes system on* (a, b). *Let nonnegative integers* k, l, m, *and* α *be given satisfying* $l + m = k$, $1 \leqslant \alpha - l$, *and* $\alpha + m \leqslant N$. *Suppose that* $a < x_1 < \cdots < x_k < b$

$$p = u_\alpha + \sum_{i=1}^{k} a_i u_{\lambda_i}, \qquad q = u_\alpha + \sum_{i=1}^{k} b_i u_{\lambda_i},$$

and

$$0 = p(x_i) = q(x_i)$$

where $1 \leqslant \gamma_i \leqslant \lambda_i < \alpha$ *for* $i = 1, \ldots, l$ *and* $\alpha < \lambda_i \leqslant \gamma_i \leqslant N$ *for* $i = l + 1, \ldots, k$. *Then* $|p(x)| \leqslant |q(x)|$ *for all* $x \in (a, b)$ *with strict inequality if* $x \neq x_i$, $i = 1, \ldots, k$ *provided* $p \neq q$.

The proof of this theorem may be safely omitted since it is quite similar to the proof of Theorem 1.

3. Applications. Throughout this section we will assume that $\{u_i\}_{i=1}^N \subset C(a, b)$ is a Descartes system on (a, b) and that $\{u_i\}_{i=1}^N \subset L_p(a, b)$, $1 \leqslant p \leqslant \infty$, whenever the L_p norm is discussed (when $p = \infty$ we really mean $C[a, b]$ with the supremum norm and the $\{u_i\}_{i=1}^N$ from a Descartes system on the closed interval $[a, b]$). We will denote by $\|f\|_p$ the integral $(\int_a^b |f(t)|^p\, dt)^{1/p}$ with appropriate modification if $p = \infty$.

Let $1 \leqslant k \leqslant N$ and let $\Lambda = \{\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$: λ_i are integers and $1 \leqslant \lambda_1 < \cdots < \lambda_k < N\}$. For any $\boldsymbol{\lambda} \in \Lambda$ we set $S(\boldsymbol{\lambda}) = \operatorname{span}\{u_{\lambda_i}\}_{i=1}^k$. Finally, $d_p(\boldsymbol{\lambda})$ will denote the $L_p(a, b)$ distance of u_N from $S(\boldsymbol{\lambda})$ (i.e. $d_p(\boldsymbol{\lambda}) = \inf\{\|u_N - s\|_p : s \in S(\boldsymbol{\lambda})\}$).

The next theorem was proved in [1] for $p = \infty$ under slightly more restrictive hypotheses and for $1 \leqslant p \leqslant \infty$ by Pinkus [5]. We present a different proof using Theorem 1.

THEOREM 3. *Let* k *and* N *be as above and* $\boldsymbol{\lambda} \in \Lambda$. *Set* $\boldsymbol{\lambda}^* = (N-k, \ldots, N-1)$, *then if* $\boldsymbol{\lambda} \neq \boldsymbol{\lambda}^*$

$$d_p(\boldsymbol{\lambda}^*) < d_p(\boldsymbol{\lambda})$$

for $1 \leqslant p \leqslant \infty$.

PROOF. Let $s_{\boldsymbol{\lambda}}$ be the best L_p approximation to u_N from $S(\boldsymbol{\lambda})$ with $\lambda \neq \lambda^*$. Then $q = u_N - s_{\boldsymbol{\lambda}}$ has k distinct zeroes in (a, b), say $x_1, \ldots, x_k$ **[4]**. Determine $s^* \in S(\boldsymbol{\lambda}^*)$ by the equations $(u_N - s^*)(x_i) = 0$ for $i = 1, \ldots, k$. These equations are uniquely solvable by (2.1). Theorem 1 now implies that

$$|(u_N - s^*)(x)| < |(u_N - s)(x)|$$

for all $x \in (a, b)$, $x \neq x_i$, $i = 1, \ldots, k$. Thus for $1 \leqslant p < \infty$ $\|u_N - s^*\|_p < \|u_N - s\|_p$. If $p = \infty$ we have $\|u_N - s^*\|_\infty \leqslant \|u_N - s\|_\infty$, but the additional assumption that the $\{u_i\}_{i=1}^N$ are a Descartes system on $[a, b]$ then yields the strict inequality.

We may obtain a similar result by using Theorem 2 as follows. Let α be an integer between 1 and N, and let l, m, and k be given nonnegative integers satisfying $1 \leqslant \alpha - l, \alpha + m \leqslant N$, and $l + m = k$. We set $\Lambda(l, m; \alpha) = \{\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)\colon\ 1 \leqslant \lambda_1 < \cdots < \lambda_l < \alpha < \lambda_{l+1} < \cdots < \lambda_k \leqslant N,\ \lambda_i$ integers$\}$. For $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ in $\Lambda(l, m; \alpha)$ we say $\boldsymbol{\lambda} \leqslant \boldsymbol{\mu}$ provided

(i) $\lambda_i \leqslant \mu_i$, $i = 1, \ldots, l$, and

(ii) $\lambda_i \geqslant \mu_i$, $i = l + 1, \ldots, k$.

Thus the "largest" element in $\Lambda(l, m; \alpha)$ is $\boldsymbol{\lambda}^{**} = (\alpha - l, \alpha - l + 1, \ldots, \alpha - 1, \alpha + 1, \ldots, \alpha + m)$. With this notation we can now state

THEOREM 4. *Let* $\boldsymbol{\lambda} \in \Lambda(l, m; \alpha)$ *with* $\boldsymbol{\lambda} \neq \boldsymbol{\lambda}^{**}$. *Then* $d_p(\boldsymbol{\lambda}^{**}) < d_p(\boldsymbol{\lambda})$ *for* $1 \leqslant p \leqslant \infty$.

This theorem is proved in a manner analogous to Theorem 3.

The conjecture of Lorentz is a corollary of Theorem 3 since $\{x^i\}_{i=0}^N$ is a Descartes system on $(0, \infty)$. More generally, if one wants to approximate x^α, $1 \leqslant \alpha \leqslant N$ with α an integer, on $0 < a < b < \infty$ in the $L_p[a, b]$ norm by linear combinations of the form $\sum_{i=1}^k a_i x^{\lambda_i}$ where k is fixed, $\lambda_i \neq \alpha$, $i = 1, \ldots, k$ and λ_i nonnegative integers, then Theorem 4 tells us that the optimal set $\{\lambda_i^*\}_{i=1}^k \equiv B$ must satisfy $\{B \cup \alpha\}$ is a set of consecutive integers.

We remark in closing that Theorems 3 and 4 could be strengthened to include approximation in $L_p(\mu)$ where μ is a positive measure such that span $\{u_i\}_{i=1}^N$ is of dimension N in $L_p(\mu)$. This is easy to see since Theorems 1 and 2 are pointwise results.

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