PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 70, Number 1, June 1978

A MOMENT PROBLEM ON JORDAN DOMAINS

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ABSTRACT. Let D_1, D_2 be Jordan domains on the complex z-plane such that $\int_{D_1} z^n\,dm = \int_{D_2} z^n\,dm$ for every nonnegative integer n. Here m denotes two-dimensional Lebesgue measure. Does it follow that $D_1 = D_2$? This moment problem on Jordan domains was posed by H. S. Shapiro [2, p. 193, Problem 1]. In this paper we construct a counterexample and study conditions on D_1 and D_2 which imply that the above equality does not hold for some n.

1. A counterexample. We first construct a counterexample. Let $R_1 = \{z \mid 3 < |z+1| < \sqrt{10}\}$, $R_2 = \{z \mid 3 < |z-1| < \sqrt{10}\}$, $E = R_1 \cap R_2 \cap \{z \mid \operatorname{Im} z > 0\}$ and $F = [\{\Delta_3(-1) \cap \Delta_3(1)\} - \{\Delta_1(-1) \cup \Delta_1(1)\}] \cap \{z \mid \operatorname{Im} z > 0\}$, where $\Delta_r(c)$ denotes the open disc with radius r and center at c. Set $D_1 = \{R_1 \cup \Delta_1(1) \cup \bar{F}\}^\circ - \bar{E}$ and $D_2 = \{R_2 \cup \Delta_1(-1) \cup \bar{F}\}^\circ - \bar{E}$, where $\bar{F}$ denotes the closure of F and A° denotes the interior of A (Figure 1, D_2 is the reflection of D_1 in the imaginary axis). Since

$$\int_{R_1 \cup \Delta_1(1)} z^n\,dm = \pi\{(-1)^n + 1^n\} = \int_{R_2 \cup \Delta_1(-1)} z^n\,dm,$$

two distinct Jordan domains D_1 and D_2 satisfy

$$\int_{D_1} z^n\,dm = \int_{D_2} z^n\,dm$$

for every nonnegative integer n.

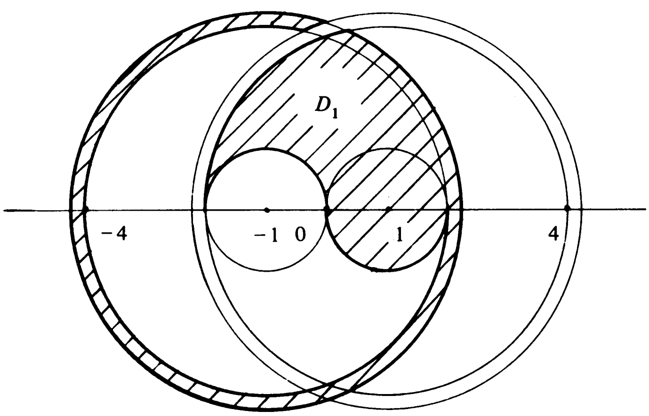

FIGURE 1

Received by the editors September 9, 1977 and, in revised form, October 6, 1977.
AMS (MOS) subject classifications (1970). Primary 30A80; Secondary 30A82.
Key words and phrases. Jordan domains, moment problems, polynomial approximation.

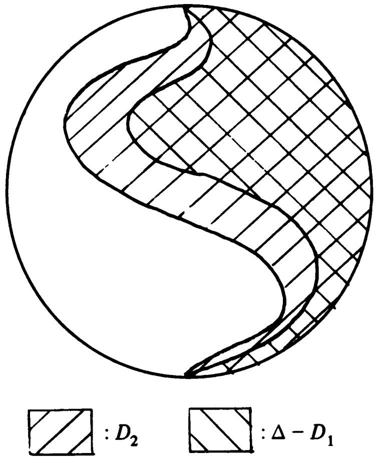

FIGURE 2

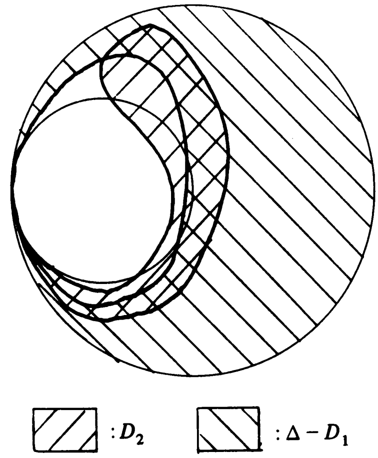

FIGURE 3

REMARK. By deforming F in the above counterexample we can construct another counterexample with D_1 and D_2 which are not congruent.

2. Conditions. We next summarize the known results.

PROPOSITION 1. *Let D_1, D_2 be Jordan domains satisfying one of the following conditions*:

(i) *D_1 and D_2 are distinct and there is an analytic L^1 function ϕ on a Jordan domain containing $D_1 \cup D_2$ such that* $\operatorname{Re} \phi(z) > 0$ *on $D_1 - D_2$ and* $\operatorname{Re} \phi(z) < 0$ *on $D_2 - D_1$.*

(ii) *$\overline{D}_1$ and $\overline{D}_2$ are disjoint or intersect in just one point.*

(iii) *We denote by γ the boundary of the unbounded complementary component of $\overline{D}_1 \cup \overline{D}_2$ and denote by D_i^e the exterior of D_i. The restriction of γ to $D_1^e \cup D_2^e$ is not analytic.*

Then

$$\int_{D_1} z^n \, dm \neq \int_{D_2} z^n \, dm$$

for some nonnegative integer n.

Condition (i) is very useful. Condition (ii) was given by H. S. Shapiro **[2]** (see also **[1]**) and condition (iii) was given implicitly by D. Aharonov and H. S. Shapiro **[1**, Lemmas 2.2 and 6.1**]** (see also P. J. Davis **[4**, p. 21**]**).

Finally we give our new condition. To do so, we recall the following proposition proved in **[6]**.

PROPOSITION A. *Let D be a domain containing the origin* 0. *Let ν be an L^1 function on* $\mathbf{C}$ *such that $\nu(z) \geqslant k$ a.e. on D for a positive number k and $\nu(z) = 0$ a.e. on the complement of D. If $f'(0) = \int_D f'\nu \, dm / \int \nu \, dm$ for every analytic function f on D such that $\int_D |f'|^2 \nu \, dm < \infty$, then $D \subset \Delta_r(0)$, where $r = (\int \nu \, dm / k\pi)^{1/2}$. The equality* $\sup_{z \in D} |z| = r$ *holds if and only if $\nu(z) = k$ a.e. on D and $D = \Delta_r(0) - E$, where E is a relatively closed subset of $\Delta_r(0)$*

such that $E \cap K$ is removable with respect to analytic functions with finite Dirichlet integrals for every compact subset K of $\Delta_r(0)$.

By using this proposition we have

PROPOSITION 2. *Let D_1, D_2 be two distinct Jordan domains. Suppose there is an open disc $\Delta = \Delta_r(c)$ having the following properties:*

(i) $D_1 \cup D_2 \subset \Delta$.

(ii) $\chi_{D_2 \cup (\Delta - D_1)} = \chi_\Omega$ *a.e. on* $\mathbf{C}$ *for a simply connected domain* Ω *such that*

(a) $c \in \Omega$,

(b) *every analytic L^2 function on Ω can be approximated arbitrarily closely in the L^2 norm by a sequence of polynomials, where χ_Ω denotes the characteristic function of Ω.*

Then

$$\int_{D_1} z^n \, dm \neq \int_{D_2} z^n \, dm$$

for some nonnegative integer n.

PROOF. Assume that $\int_{D_1} z^n \, dm = \int_{D_2} z^n \, dm$ for every n and set $\nu(z) = \chi_\Omega(z) + \chi_{D_2 - D_1}(z)$. Then

$$\begin{aligned} \int_\Omega z^n \nu(z) \, dm &= \int_{D_2 \cup (\Delta - D_1)} z^n \, dm + \int_{D_2 \cap (\Delta - D_1)} z^n \, dm \\ &= \int_\Delta z^n \, dm + \int_{D_2} z^n \, dm - \int_{D_1} z^n \, dm = c^n m(\Delta). \end{aligned} \tag{1}$$

Since every analytic L^2 function on Ω can be approximated by polynomials, we have

$$f(c) = \int_\Omega f\nu \, dm \Big/ \int \nu \, dm$$

for every analytic L^2 function on Ω. From the definition of $\nu(z)$, it follows that $\nu(z) \geqslant 1$ a.e. on Ω, $\nu(z) = 0$ a.e. on the complement of Ω, $\int \nu \, dm = \pi r^2$ and $\sup_{z \in \Omega} |z - c| = r$. Hence, by Proposition A, we have $\chi_{D_2 - D_1}(z) = 0$ a.e. on $\mathbf{C}$. Since D_1 and D_2 are Jordan domains satisfying $m(D_1) = m(D_2)$, we have $D_1 = D_2$. This is a contradiction.

Proposition 2 is applicable to the case treated in Problem 4 of [2] (Figure 2, cf. Remark). In this case Ω is a Jordan domain, and so Ω has an approximation property mentioned in (b) of Proposition 2. Carathéodory domains also have this property; Ω is called a Carathéodory domain if its boundary coincides with the boundary of the unbounded complementary component of its closure. There are non-Carathéodory domains having this property (see S. N. Mergelyan [5] and J. E. Brennan [3]). Proposition 2 is also applicable to these cases (Figure 3).

REMARK. If Ω is a Carathéodory domain such that Ω^e is connected, for example, we can omit (a) of Proposition 2. In fact, for every $\zeta \in \Omega^e$, $1/(z - \zeta)$ can be approximated uniformly on $\overline{\Omega} \cup \{c\}$ by polynomials.

Hence, from (1), we have

$$\hat{\nu}(\zeta) = m(\Delta)/(c - \zeta),$$

where $\hat{\nu}$ denotes the Cauchy transform of ν. Since $\hat{\nu}$ is continuous, it follows that $c \in \Omega$.

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