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ON THE OPERATOR EQUATION $AX + XB = Q$

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ABSTRACT. Consider the operator equation (*) $AX + XB = Q$; here A and B are (possibly unbounded) selfadjoint operators and Q is a bounded operator on a Hilbert space. The theory of one parameter semigroups of operators is applied to give a quick derivation of M. Rosenblum's formula for approximate solutions of (*). Sufficient conditions are given in order that (*) has a solution in the Schatten-von Neumann class $\mathcal{C}_p$ if Q is in $\mathcal{C}_p$. Finally a sufficient condition for solvability of (*) is given in terms of T. Kato's notion of smoothness.

1. Introduction. Suppose A and B are (possibly unbounded) selfadjoint operators on a complex separable Hilbert space $\mathcal{H}$. Of concern is the operator equation

$$AX + XB = Q \tag{1}$$

where Q is a given bounded operator. By a solution of (1) we mean a bounded operator X on $\mathcal{H}$ which maps $\mathfrak{D}(B)$ (= the domain of B) into $\mathfrak{D}(A)$ such that

$$AXf + XBf = Qf$$

holds for all f in $\mathfrak{D}(B)$.

Marvin Rosenblum [7] has studied (1) by a perturbation procedure. For other papers on the subject see the bibliographies in [5], [7]. We shall derive Rosenblum's formula ((4) below) for approximate solutions of (1) as an elementary consequence of the easy parts of the Hille-Yosida-Phillips theory of semigroups of operators. As a byproduct of this approach we find a simple sufficient condition for (1) to have a solution in the Schatten-von Neumann class $\mathcal{C}_p$ of compact operators when Q belongs to $\mathcal{C}_p$, $1 \leqslant p \leqslant \infty$.

2. The main result. The Schatten-von Neumann class $\mathcal{C}_p$ of operators on $\mathcal{H}$ is the set of all compact operators L on $\mathcal{H}$ for which $|||L|||_p < \infty$ where

$$|||L|||_p^p = \left(\text{trace } |L|^P\right) \quad \text{for } 1 \leqslant p < \infty,$$
$$|||L|||_\infty = \|L\| = \text{the operator norm of } L;$$

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here $|L| = (L^*L)^{1/2}$. $\mathcal{C}_p$ is a Banach space under $|||\cdot|||_p$ (cf. e.g. **[6]**, **[8]**). $\mathcal{C}_\infty$ is the set of all compact operators on $\mathcal{H}$.

THEOREM. *Let A, B be selfadjoint operators on $\mathcal{H}$ and let Q be bounded. The approximation equation*

$$iyX_y + AX_y + X_yB = Q \tag{2}$$

has for $y > 0$ a unique solution given by the weak integrals (with $z = x + iy$)

$$X_y = -i\int_0^\infty e^{-2ty}\exp(itA)Q\exp(itB)\,dt \tag{3}$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty}(A - \bar{z}I)^{-1}Q(B - zI)^{-1}\,dx. \tag{4}$$

There is a bounded solution of (1) *iff* $\{\|X_y\|: 0 < y < 1\}$ *is bounded. Let* $1 \leqslant p \leqslant \infty$. *There is a solution of* (1) *in* $\mathcal{C}_p$ *if*

(i_p) $\{|||X_y|||_p: 0 < y < 1\}$ *is bounded,*

(ii) *for each $\varepsilon > 0$ there is a finite dimensional subspace M_ε of $\mathcal{H}$ such that if Z_y is the restriction of X_y to $M_\varepsilon^\perp$ then $\|Z_y\| \leqslant \varepsilon$ for $0 < y < 1$.*

PROOF. Let $\langle\cdot,\cdot\rangle$ denote the $L^2(\mathbf{R})$ inner product. Saying that $X = \int_{-\infty}^{\infty} R(t)\,dt$ (weak integral) means that for all $f, g \in \mathcal{H}$, the complex-valued function $t \to \langle R(t)f, g\rangle$ is integrable and $\langle Xf, g\rangle = \int_{-\infty}^{\infty}\langle R(t)f, g\rangle\,dt$.

Define G on $\mathcal{C}_p$ ($1 \leqslant p \leqslant \infty$) as follows: for $X, Y \in \mathcal{C}_p$, $X \in \mathfrak{D}(G)$ and $GX = Y$ means that $X(\mathfrak{D}(B)) \subset \mathfrak{D}(A)$ and for all $f \in \mathfrak{D}(B)$, $AXf + XBf = Yf$. Then iG generates a strongly continuous (or (C_0)) group of isometries (cf. [2], [3], [10]) on $\mathcal{C}_p$ given by

$$\exp(itG)(X) = \exp(itA)X\exp(itB), \qquad X \in \mathcal{C}_p.$$

The proof is straightforward; for details see **[1]**, which also contains the converse result for $p \neq 2$.

Recall that if C generates a (C_0) contraction semigroup $\{\exp(tC)\}$, then for all λ with Re $\lambda > 0$, λ is in the resolvent set of C and

$$(\lambda I - C)^{-1} = \int_0^\infty e^{-\lambda t}\exp(tC)\,dt. \tag{5}$$

Consequently for $Q \in \mathcal{C}_p$ and μ such that Im $\mu > 0$,

$$\begin{aligned}(\mu I + G)^{-1}Q &= -i\int_0^\infty e^{i\mu t}\exp(itG)Q\,dt \\ &= -i\int_0^\infty e^{i\mu t}\exp(itA)Q\exp(itB)\,dt.\end{aligned} \tag{6}$$

Also, $X = (\mu I + G)^{-1}Q$ satisfies

$$\mu X + AX + XB = Q.$$

Taking $\mu = 2iy$ with $y > 0$ and writing X_y for X, we find that the unique solution of (2) is given by (3). Also, (5) (with $C = iG$) gives the easy estimate

$$|||X_y|||_p \leqslant |||Q|||_p/2y < \infty.$$

Let $j(t) = e^{-yt}L\exp(itC)$ for $t \geqslant 0$ and $j(t) = 0$ for $t < 0$, where L is bounded and C selfadjoint. Using (5) we compute the Fourier transform of j to be

$$\hat{j}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} j(t)\, dt = \frac{i}{\sqrt{2\pi}} (iy + s + C)^{-1}.$$

Let

$$k(t) = e^{-yt}\exp(-itA), \qquad h(t) = e^{-yt}Qe^{itB}$$

for $t \geqslant 0$, and let $k(t) = h(t) = 0$ for $t < 0$. Taking successively $j = h$ ($L = Q$, $C = B$) and $j = k$ ($L = I$, $C = -A$) and plugging into the Plancherel formula $\langle h, k \rangle = \langle \hat{h}, \hat{k} \rangle$ we conclude that the expressions in (3) and (4) are equal. These formulas for X_y were derived assuming Q to be compact, but it is straightforward to check that they are valid whenever Q is bounded. That (1) has a bounded solution iff $\{\|X_y\| : 0 < y < 1\}$ is bounded follows easily as in [7].

Formula (4) for X_y is due to Rosenblum [7]; formula (3) appears to be new.

Next we ask: When is there a solution of (1) in $\mathcal{C}_p$? First we deal with the $\mathcal{C}_\infty$ case.

Lemma. *Let $\{T_n\}$ be a sequence of operators in $\mathcal{C}_\infty$. Then $\{T_n\}$ is precompact in $\mathcal{C}_\infty$ iff the following two conditions hold.*

(i) *$\{\|T_n\|\}$ is a bounded sequence.*

(ii) *For each $\varepsilon > 0$ there is a finite dimensional subspace M_ε of $\mathcal{H}$ such that if S_n is the restriction of T_n to $M_\varepsilon^\perp$, then $\|S_n\| \leqslant \varepsilon$ for all n.*

The straightforward proof is omitted.

The $\mathcal{C}_\infty$ assertion of the Theorem now follows; any limit of a sequence X_{y_n} with $y_n \downarrow 0$ is a solution of (1). Next let $1 \leqslant p < \infty$ and replace (i) of the Lemma by

(i_p) $\{|||T_n|||_p\}$ is a bounded sequence.

(i_p) and (ii) do *not* imply that $\{T_n\}$ is precompact in $\mathcal{C}_p$ for $p < \infty$. However, since (i_p) implies (i), (i_p) and (ii) imply $\|T_{n_k} - T\| \to 0$ for some $T \in \mathcal{C}_\infty$ and some subsequence $\{T_{n_k}\}$ by the Lemma. Let $R_k = |T_{n_k}|^p$. Then for all finite rank operators L on $\mathcal{H}$,

$$\left|\operatorname{trace}\left(|T|^p L\right)\right| = \lim_{k \to \infty} |\operatorname{trace}(R_k L)| \leqslant K_0 \|L\|$$

where $K_0 = \sup_k |||R_k|||_p^p < \infty$. It follow that $|T|^p$ is in the trace class, whence $T \in \mathcal{C}_p$. Thus any limit point of X_y (with $y \downarrow 0$) is a solution of (1) which belongs to $\mathcal{C}_p$. Q.E.D.

3. Remarks. Let L be bounded and C selfadjoint. Following T. Kato [4], we say that L is C-smooth if there is a constant $k = k(L, C) > 0$ such that

$$\int_{-\infty}^{\infty} \|L \exp(itC) f\|^2\, dt \leqslant k \|f\|^2$$

for all $f \in \mathcal{H}$. This condition is of fundamental importance in scattering theory.

COROLLARY. *Let A and B be selfadjoint. Let $Q = Q_1^* Q_2$ where Q_1 is A-smooth and Q_2 is B-smooth. Then* (1) *has a solution.*

PROOF. Define S by

$$S = \int_0^\infty \exp(itA) Q_1^* Q_2 \exp(itB)\, dt$$

(weak integral). Let $f, g \in \mathcal{H}$. Then by the Schwarz inequality,

$$\begin{aligned} |\langle Sf, g\rangle|^2 &= \left| \int_0^\infty \langle Q_2 \exp(itB) f, Q_1 \exp(-itA) g \rangle dt \right|^2 \\ &\leqslant \left(\int_0^\infty \|Q_2 \exp(itB) f\|^2 dt \right) \left(\int_0^\infty \|Q_1 \exp(-itA) g\|^2 dt \right) \\ &\leqslant k(Q_2, B) k(Q_1, A) \|f\|^2 \|g\|^2. \end{aligned}$$

Thus S is bounded. Moreover, the above argument shows that $\|X_y\| \leqslant k(Q_2, B)k(Q_1, A) < \infty$ for $0 < y < 1$ (see (3)). The Corollary now follows from the Theorem.

Let Δ be the selfadjoint realization of the Laplacian on $L^2(\mathbf{R}^3)$, let a, b be nonzero real numbers, and let $A = a\Delta$, $B = b\Delta$. Let Q be the operation of multiplication by a complex-valued function V on $\mathbf{R}^3$ where $V \in L^\infty(\mathbf{R}^3) \cap L^{3/2}(\mathbf{R}^3)$. Then (see [**4**, p. 276]) by the Corollary, there is a bounded operator X such that $a\Delta Xf + bX\Delta f = Vf$ for all f in the Sobolev space $H^2(\mathbf{R}^3)$ $(= \mathfrak{D}(\Delta))$.

Our techniques extend easily to solve certain equations of the type (1) where A and B generate uniformly bounded groups on a Banach space. When the space is a complex Hilbert space, then iA and iB are *similar* to selfadjoint operators, according to a theorem of Sz.-Nagy [**9**], but the similarity transforms need not commute with one another.

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