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THE FREDHOLM RADIUS OF A BUNDLE OF CLOSED LINEAR OPERATORS

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ABSTRACT. Given a bundle of linear operators $T - \lambda S$, where T is closed and S is bounded, a sequence $\{\delta_m(T : S)\}$ of extended real numbers is defined. If T is a Fredholm operator, the limit $\lim \delta_m(T : S)^{1/m}$ exists and is equal to the supremum of all $r > 0$ such that $T - \lambda S$ is a Fredholm operator for $|\lambda| < r$.

Throughout this paper X and Y are complex Banach spaces, T is a closed linear operator with domain $D(T)$ in X and range $R(T)$ in Y, and S is a bounded linear operator from X into Y. $K(X)$ is the space of compact linear operators on X, and $\Phi(T : S)$ is the set of those complex numbers λ for which $T - \lambda S$ is a Fredholm operator.

Given $m \geqslant 1$, the element $(x_1, \ldots, x_m)$ of $D(T)^m$ is called a chain for T and S if $Tx_i = Sx_{i-1}$ for $i = 2, \ldots, m$. Put

$$\delta_m = \delta_m(T : S) = \sup_{C \in K(X)} \inf_{(x_1, \ldots, x_m)} \frac{\|Tx_1\|}{\|(I - C)x_m\|},$$

where the infimum is taken over all chains $(x_1, \ldots, x_m)$ for T and S. Here I denotes the identity mapping in X.

When $X = Y$ and $S = I$, the chains for T and S are of the form $(T^{m-1}x, \ldots, Tx, x)$ with $x \in D(T^m)$, and

$$\delta_m(T : I) = \delta_1(T^m : I) = \sup_{C \in K(X)} \inf_{x \in D(T)} \frac{\|T^m x\|}{\|(I - C)x\|}.$$

Roughly speaking $\delta_1(T : I)$ is the reduced minimum modulus of T corresponding to the m-seminorm introduced by A. Lebow and M. Schechter in [4]. $\delta_1(T : I)$ was studied in [5] and there it was shown that for a Fredholm operator T, $\lim \delta_1(T^m : I)^{1/m}$ exists and is equal to the distance $d(0, \mathbf{C} \setminus \Phi(T : I))$ of 0 to the complement of the Fredholm set of T.

THEOREM. *Let T be a Fredholm operator. Then*

$$\lim_{m \to \infty} \delta_m(T : S)^{1/m}$$

exists and is equal to $d(0, \mathbf{C} \setminus \Phi(T : S))$, the Fredholm radius of T and S.

This result is closely related to the stability radius of a bundle of operators

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studied by H. Bart and D. C. Lay [1] in general and earlier by K.-H. Förster and M. A. Kaashoek [2] in the case $X = Y$ and $S = I$. In addition, the proof of the theorem requires a modification of the decomposition theorem of T. Kato [3, Theorem 4]. Both stability radius and decomposition use the following notation (see [3], [1]).

Define subspaces $N_m = N_m(T : S)$ and $R_m = R_m(T : S)$ of X by

$$N_0 = (0), \quad N_m = T^{-1}SN_{m-1},$$
$$R_0 = X, \quad R_m = S^{-1}TR_{m-1}, \qquad m = 1, 2, \ldots.$$

The smallest number m for which the null space $N(T) = N_1$ of T is not contained in R_m will be denoted by $\nu(T : S)$. Further let $\Delta(T : S)$ denote the open set of all λ in $\Phi(T : S)$ such that $\nu(T - \lambda S : S) = \infty$.

1.1. *Stability radius* [1]. For $m \geqslant 1$ let $\gamma_m = \gamma_m(T : S)$ denote the supremum of all $c \geqslant 0$ with the property that

$$\|Tx_1\| \geqslant c \cdot d(x_m, N_m)$$

for every chain $(x_1, \ldots, x_m)$. If T is a Fredholm operator, the limit $\lim \gamma_m(T : S)^{1/m}$ exists and is equal to $d(0, \mathbf{C} \setminus [\Delta(T : S) \cup \{0\}])$, i.e. the supremum of all $r > 0$ such that $n(T - \lambda S) = \dim N(T - \lambda S)$ and $d(T - \lambda S) = \operatorname{codim} R(T - \lambda S)$ are constant on $0 < |\lambda| < r$.

1.2. Suppose $n(T)$ is finite. Then for $m \geqslant 1$ there is a compact projection P_m of X onto N_m such that $\|P_m\| \leqslant m \cdot n(T)$. Then

$$\|(I - P_m)x_m\| \leqslant \|I - P_m\| d(x_m, N_m) \leqslant \|I - P_m\| \gamma_m(T : S)^{-1} \|Tx_1\|$$

for every chain $(x_1, \ldots, x_m)$ where, as usual, $0^{-1} = \infty$. Therefore, γ_m and δ_m are related by $\gamma_m \leqslant (1 + mn(T))\delta_m$.

2.1. *Decomposition* [3, *Theorem* 4]. Let T be a Fredholm operator such that $\nu(T : S)$ is finite. There exist topological decompositions $X = X_0 \oplus X_1$ and $Y = Y_1 \oplus Y_1$ which completely reduce T and S. For $i = 0, 1$, let T_i and S_i denote the restrictions of T and S to X_i viewed as operators into Y_i. Then $\nu(T_0 : S_0) = \infty$, S_1 is bijective, $S_1^{-1}T_1$ is nilpotent, and $\dim X_1 = \dim Y_1 < \infty$. As a consequence, we have $\Delta(T_0 : S_0) = \Delta(T : S) \cup \{0\}$.

2.2. Now take $0 < \rho < d(0, \mathbf{C} \setminus (T : S))$ and let Δ_ρ denote the set of all complex numbers λ such that $|\lambda| \leqslant \rho$ and $\nu(T - \lambda S : S) < \infty$. By induction we remove the finite set Δ_ρ from $\Delta(T : S)$ and obtain decompositions $X = X_\rho \oplus X_1$ and $Y = Y_\rho \oplus Y_1$ such that $\Delta(T_\rho : S_\rho) = \Delta(T : S) \cup \Delta_\rho$, where T_ρ and S_ρ are restrictions of T and S to X_ρ as in 2.1.

2.1 and 2.2 remain true if Fredholm operators are replaced by semi-Fredholm operators. However, this is not possible in the case 1.1, see [1, 4.1].

Now we are able to prove the theorem. It will be shown

(a) $d(0, \mathbf{C} \setminus \Phi(T : S)) \leqslant \liminf \delta_m(T : S)^{1/m}$ and

(b) $\limsup \delta_m(T : S)^{1/m} \leqslant d(0, \mathbf{C} \setminus \Phi(T : S))$.

Both parts together establish the theorem.

(a) Since T is Fredholm, $d(0, \mathbf{C} \setminus \Phi(T : S))$ is positive. Take $0 < \rho < d(0, \mathbf{C} \setminus \Phi(T : S))$. 1.1 and 2.2 imply that $\rho \leqslant \lim \gamma_m(T_\rho : S_\rho)^{1/m}$. For $m =$

1, 2, ... let P_m be a projection of X onto $N_m(T_\rho : S_\rho)$ with $\|P_m\| \leq mn(T_\rho)$ and let $(x_1, \ldots, x_m)$ be a chain for T and S. Furthermore let P and Q be the bounded projections of X onto X_1 along X_ρ and of Y onto Y_1 along Y_ρ, respectively. Then $P_m P = 0$, $P_m + P \in K(X)$, and it is easy to verify that $((I - P)x_1, \ldots, (I - P)x_m)$ is a chain for T_ρ and S_ρ. But then

$$\begin{aligned} \|[I - (P_m + P)]x_m\| &= \|(I - P_m)(I - P)x_m\| \\ &\leq \|I - P_m\| \gamma_m(T_\rho : S_\rho)^{-1} \|T_\rho(I - P)x_1\| \\ &\leq \|I - P_m\| \gamma_m(T_\rho : S_\rho)^{-1} \|I - Q\| \, \|Tx_1\|, \end{aligned}$$

as in 1.2. Hence

$$\left[(1 + mn(T_\rho))\|I - Q\|\right]^{-1} \gamma_m(T_\rho : S_\rho) \leq \delta_m(T : S),$$

and consequently

$$\rho \leq \lim \gamma_m(T_\rho : S_\rho)^{1/m} \leq \liminf \delta_m(T : S)^{1/m},$$

which proves (a).

(b) Take $0 < |\lambda| < \alpha < \limsup \delta_m(T : S)^{1/m}$. First, suppose $\nu(T : S) = \infty$. This restriction will be removed later with the aid of 2.1. It will be shown that $T - \lambda S$ is a Fredholm operator. There exists some m and a compact operator $C = C_{\alpha,m}$ on X such that

$$\|(I - C)x_m\| \leq \alpha^{-m} \|Tx_1\|$$

for every chain $(x_1, \ldots, x_m)$ for T and S. Since T is Fredholm and $\nu(T : S) = \infty$, that is, $N(T) \subseteq R_{m-1}$, there exists a relative inverse L_m of T such that $L_m T R_n \subseteq R_n$ for $n = 0, 1, \ldots, m - 1$. Since TR_{m-1} has finite deficiency in Y, there is a projection Q of Y onto TR_{m-1} such that $\|Q\| \leq 1 + md(T)$. Take $y \in Y$ and put

$$x_i = (L_m S)^{i-1} L_m Qy, \qquad i = 1, \ldots, m.$$

$(x_1, \ldots, x_m)$ turns out to be a chain with $Tx_1 = Qy$. Consequently

$$\|(I - C)(L_m S)^{m-1} L_m Q\| \leq \alpha^{-m} \|Q\|,$$

and taking $Q = I - P$, $P \in K(Y)$, we have

$$\|(I - C)(L_m S)^{m-1} L_m (I - P)S\| = \|(L_m S)^m - K_m\| \leq \alpha^{-m} \|Q\| \, \|S\|$$

with some $K_m \in K(X)$.

Now let π denote the canonical mapping from $B(X)$ onto $B(X)/K(X)$. Here $B(X)$ is the space of all bounded linear operators on X. Take any relative inverse L of T. Then $L_m - L$ is degenerate, hence $\pi(L_m S) = \pi(LS)$, and the last inequality reads

$$\|\pi(LS)^m\| \leq \alpha^{-m}(1 + md(T))\|S\|.$$

This implies $r_\sigma \leq \alpha^{-1} < |\lambda|^{-1}$.

Here r_σ is the spectral radius of $\pi(LS)$. But then $\lambda^{-1}\pi(I) - \pi(LS)$ is invertible in $B(X)/K(X)$, hence $I - \lambda LS$ is Fredholm and so is $T - \lambda TLS$.

Since L is a relative inverse of T, $TL = I - R$, where R is a compact projection. So $T - \lambda S$ is a Fredholm operator if $\nu(T : S) = \infty$. Now suppose $\nu(T : S)$ is finite. Then $\nu(T_0 : S_0) = \infty$ by 2.1. Let P_0 be the bounded projection of X onto X_0 along X_1. Starting with a chain for T_0 and S_0 we obtain $\delta_m(T : S) \leqslant \|P_0\|\delta_m(T_0 : S_0)$. Therefore

$$0 < |\lambda| < \limsup \delta_m(T : S)^{1/m} \leqslant \limsup \delta_m(T_0 : S_0)^{1/m},$$

and by the preceding argument $T_0 - \lambda S_0$ is Fredholm. Since $S_1^{-1}T_1$ is nilpotent and $\lambda \neq 0$, $T_1 - \lambda S_1$ is bijective, thus $T - \lambda S$ is Fredholm. This proves (b).

Corollary 1. *Let T be a Fredholm operator. Then $\Phi(T : S) = \mathbf{C}$ if and only if $\lim \delta_m^{1/m} = \infty$, i.e. if and only if for each $\varepsilon > 0$ and sufficiently large m there are compact operators $C_{\varepsilon,m}$ on X, such that for every chain $(x_1, \ldots, x_m)$*

$$\|x_m\| \leqslant \varepsilon^m \|Tx_1\| + \|C_{\varepsilon,m}x_m\|.$$

Let $\hat{X}$ be $D(T)$ endowed with the graph norm $\|x\|_T = \|x\| + \|Tx\|$, let $\hat{T}$ and $\hat{S}$ be the operators T and S considered as maps from $\hat{X}$ into Y, and let i_T be the inclusion map of $\hat{X}$ into X. Then $\hat{X}$ is a Banach space, $\hat{T}$, $\hat{S}$, i_T are bounded, and $\hat{T} = Ti_T$, $\hat{S} = Si_T$. Put

$$\hat{\delta}_m = \hat{\delta}_m(T : S) = \sup_{K \in K(\hat{X}, X)} \inf_{(x_1, \ldots, x_m)} \frac{\|\hat{T}x_1\|}{\|(i_T - K)x_m\|},$$

where $(x_1, \ldots, x_m)$ is a chain for $\hat{T}$ and $\hat{S}$. Since $\hat{T}x_1 = Tx_1$ and $C \in K(X)$ implies $Ci_T \in K(\hat{X}, X)$, we have $\delta_m(T : S) \leqslant \hat{\delta}_m(T : S)$.

Corollary 2. *Let T be a Fredholm operator. Then $\lim \hat{\delta}_m(T : S)^{1/m} = d(0, \mathbf{C} \setminus \Phi(T : S))$ and, as a consequence $\Phi(T : S) = \mathbf{C}$, if i_T is compact.*

Proof. By the preceding remark we have $d(0, \mathbf{C} \setminus \Phi(T : S)) \leqslant \liminf \hat{\delta}_m^{1/m}$. Replacing T by $\hat{T}$, S by $\hat{S}$, and $B(X)/K(X)$ by $B(\hat{X}, X)/K(\hat{X}, X)$ in part (b) of the theorem, we obtain $\limsup \hat{\delta}_m^{1/m} \leqslant d(0, \mathbf{C} \setminus \Phi(T : S))$. If i_T is compact, then $\hat{\delta}_m = \infty$, hence the corollary.

Remark [5]. Let $X = Y$ be a complex Hilbert space, suppose $S = I$, and let T be a densely defined normal Fredholm operator. Then $d(0, \mathbf{C} \setminus \Phi(T : I)) = \delta_1 = \hat{\delta}_1$. If moreover $d(0, \mathbf{C} \setminus \Phi(T : I)) < \infty$, i.e. the Fredholm set of T is not the whole plane, then there exists a compact operator K on X such that $\delta_1(T : I) = \gamma(T - K)$, where $\gamma(T - K)$ denotes the reduced minimum modulus of $T - K$. These facts use the resolution of the identity corresponding to T.

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