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METRIC CHARACTERIZATIONS OF DIMENSION FOR SEPARABLE METRIC SPACES

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ABSTRACT. A subset B of a metric space (X, d) is called a d-bisector set iff there are distinct points x and y in X with $B = \{z: d(x, z) = d(y, z)\}$. It is shown that if X is a separable metrizable space, then $\dim(X) \leqslant n$ iff X has an admissible metric d for which $\dim(B) \leqslant n - 1$ whenever B is a d-bisector set. For separable metrizable spaces, another characterization of n-dimensionality is given as well as a metric dependent characterization of zero dimensionality.

1. Introduction. Let d be a metric for a set X. A subset B of X is called *a d-bisector set in X* iff there are distinct points x_1 and x_2 in X such that $B = \{x: d(x, x_1) = d(x, x_2)\}$. In this note we will show that if X is a separable metrizable space, then $\dim(X) \leqslant n$ iff X has an admissible totally bounded metric d such that if B is any d-bisector set in X, then $\dim(B) \leqslant n - 1$. Using this result and the machinery of [4], we observe that $t(X) = \dim(X)$ for every separable metrizable space X, where $t(X)$ is defined in the same way as in the first author's definition of the reduced bisector dimension function $r(X)$ except that only totally bounded metrics are considered.

A metric d for a set X is said to be *strongly rigid* provided that if the two element subsets $\{x_1, x_2\}$ and $\{y_1, y_2\}$ of X are distinct, then $d(x_1, x_2) \neq d(y_1, y_2)$. With respect to strongly rigid metrics, see [1], [2], [5] and [6]. We define a metric d on a set X to be *star rigid* iff whenever x, y and z are points of X with $y \neq z$, then $d(x, y) \neq d(x, z)$. Since every strongly rigid metric is star rigid, we have that if X is separable metrizable and $\dim(X) = 0$, then X has an admissible totally bounded star rigid metric [1]. We shall show below that if a nonempty space X has an admissible totally bounded star rigid metric, then $\dim(X) = 0$.

We shall use [7] as a general reference on dimension theory.

2. Theorems. Combining Lemma 3.1 and Theorem 1.2 of [4], one obtains the following: if X is a compact metrizable space with $\dim(X) = n$, then X has an admissible metric d such that if B is a d-bisector set in X, then $\dim(B) \leqslant n - 1$. Using this result, we establish the following more general theorem.

THEOREM 1. *Let X be a separable metrizable space. Then,* $\dim(X) \leqslant n$ *iff X*

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has an admissible totally bounded metric d such that if B is a d-bisector set in X, then $\dim(B) \leqslant n - 1$.

PROOF. Let $\dim(X) \leqslant n$. The space X may be embedded in a compact metrizable space Y where $\dim(Y) \leqslant n$. Choose an admissible metric ρ for Y such that if B is a ρ-bisector set in Y, then $\dim(B) \leqslant n - 1$. Let d be the metric for X obtained by restricting ρ to $X \times X$. Since ρ is a metric for a compact space, ρ is totally bounded and d, being a restriction of ρ, is also a totally bounded metric. If $A \subset X$ is a d-bisector set in X, then $A = \{x: d(x, x_1) = d(x, x_2)\}$ for a pair of points x_1 and x_2 in X. Then $A = B \cap X$ where $B = \{z \in Y: \rho(z, x_1) = \rho(z, x_2)\}$ and since $\dim(B) \leqslant n - 1$, necessarily $\dim(A) \leqslant n - 1$.

Conversely, let d be a totally bounded metric for X such that $\dim(B) \leqslant n - 1$ for every d-bisector set $B \subset X$. Let $x \in X$ and $e > 0$ be arbitrarily chosen. Our aim is to show that there is a base at x whose boundaries have dimension $\leqslant n - 1$. In case x is an isolated point the statement is trivial. In the opposite case there exist points $x_1, x_2, \dots, x_m$, none equal to x, such that the e-balls $S_e(x_i)$ centered on them cover X. Let $H_i = \{z: d(x, z) < d(z, x_i)\}$ for $i = 1, 2, \dots, m$. Note that

$$\mathrm{Bd}(H_i) \subset \{z: d(x, z) = d(z, x_i)\}$$

so that $\dim(\mathrm{Bd}(H_i)) \leqslant n - 1$. Also note that H_i is open for each i so that $H = \bigcap\{H_i: i = 1, \dots, m\}$ is an open neighborhood of x. Since $\mathrm{Bd}(H) \subset \bigcup\{\mathrm{Bd}(H_i)\}$, we have that $\dim(\mathrm{Bd}(H)) \leqslant n - 1$. Let $z \in X - S_e(x)$. There exists an x_i with $d(z, x_i) < e$; since $d(z, x) > e$, necessarily $z \notin H_i$. But then $z \notin H$, that is, $H \subset S_e(x)$. It follows that the point x has a neighborhood base consisting of open sets having boundaries of dimension less than n from which it follows that $\dim(X) \leqslant n$, completing the proof.

Using Theorem 1 and the principal result of [1] we now establish the following theorem.

THEOREM 2. *Let X be a separable metrizable space. Then,* $\dim(X) = 0$ *iff X is nonempty and has an admissible totally bounded star rigid metric.*

PROOF. Let X be a separable metrizable space with $\dim(X) = 0$. The space X is homeomorphic to a subspace S of the Cantor set C. But in [1] it was shown that the space C has an admissible strongly rigid metric ρ and the restriction of ρ to $S \times S$ yields an admissible totally bounded strongly rigid (hence star rigid) metric for S, hence for X.

Conversely, let d be an admissible totally bounded star rigid metric for the nonempty space X. If x_1 and x_2 are distinct points of X, then $\{z: d(x_1, z) = d(x_2, z)\} = \varnothing$ because d is star rigid, that is, if B is any d-bisector set in X, then $\dim(B) = -1$. By Theorem 1 it follows that $\dim(X) \leqslant 0$ and since X is nonempty, $\dim(X) = 0$, completing the proof.

Let (X, d) be a metric space. We write $Y \rhd Z$ iff $Z \subset Y \subset X$ and Z is a ρ-bisector set in Y where ρ denotes the restriction of d to $Y \times Y$. A reduced chain of length n is a chain

$$X = X_0 \rhd X_1 \rhd \cdots \rhd X_{n-1} \rhd X_n$$

such that $\dim(X_n) \leqslant 0$ and $\dim(X_{n-1}) > 0$. Let $r(X, d) = n$ if there exists a reduced chain of length n but no reduced chain of length greater than n. If there exist reduced chains of arbitrarily great length, let $r(X, d) = \infty$. Define $r(X)$ to be the minimum of $r(X, d)$ taken over the set of all metrizations of the space X. The dimension function $r(X)$ was introduced in **[4]** where it was shown that $r(X) = \dim(X)$ for every compact metrizable space X. We modify $r(X)$ for separable metrizable spaces X by letting $t(X)$ be the minimum of $r(X, d)$ taken over the set of all admissible totally bounded metrics for the space X. Using Theorem 1 of this note in place of Theorem 1.1 of **[4]**, the proof in **[4]** that $r(X) = \dim(X)$ for compact metrizable X carries over to establish the following:

THEOREM 3. *If X is a separable metrizable space, then* $t(X) = \dim(X)$.

3. Questions. In **[6]** it was shown that if X is a metrizable space with $\dim(X) = 0$ and $\operatorname{card}(X) \leqslant c$, the cardinality of the continuum, then X has an admissible strongly rigid metric. The question of whether this result remains true if we replace the condition that $\dim(X) = 0$ by the hypothesis that $\operatorname{ind}(X) = 0$ appears difficult and to our knowledge has not been answered. The following related question may be more tractable.

Question 1. If X is a metrizable space with $\operatorname{ind}(X) = 0$ and $\operatorname{card}(X) \leqslant c$, must the space X have an admissible star rigid metric?

In **[1]** it was shown that if a metrizable space X has an admissible strongly rigid metric, then $\operatorname{ind}(X) = 0$. This fact together with Theorem 2 suggests the following question.

Question 2. If a metrizable space X has an admissible star rigid metric, must $\operatorname{ind}(X) = 0$?

The dimension function r is a modification of the dimension function b of [3] and, from the definitions, it is clear that $r(X) \leqslant b(X)$ for any metrizable space X. We also have $r(X) \leqslant t(X)$ for any separable metrizable space X.

Question 3. Do any of the dimension numbers $r(X)$, $b(X)$ and $t(X)$ coincide on the class of all separable metrizable spaces?

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