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AVOIDING SELF-REFERENTIAL STATEMENTS

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ABSTRACT. Recursion-theoretic proofs of metamathematical results tend to rely on a pair of effectively inseparable r.e. sets and its properties. We establish a special property for a small configuration of such pairs and derive from it some metamathematical results not previously accessible to recursion-theoretic techniques.

0. Introduction. The applications of the dual completeness of a pair of effectively inseparable r.e. sets to metamathematical questions are manifold. Since Shepherdson 1960, however, more powerful results have been obtainable by diagonalization within a given theory. In this note, we prove a generalization of Smullyan's dual completeness result (cf. Rogers 1967, Exercise 11.29) and list some metamathematical corollaries not previously obtainable recursion-theoretically.

We let $[e]$ denote the partial recursive function with index e, and W_e the r.e. set with index e. $Texy$ is Kleene's T-predicate and, for any assertions, $\exists vRv$, $\exists vSv$, with R, S recursive, we write

$\exists vRv \preccurlyeq \exists vSv$: $\exists v[Rv \wedge \forall v' < v \neg Sv']$,

$\exists vRv \prec \exists vSv$: $\exists v[Rv \wedge \forall v' \leqslant \neg Sv']$.

A disjunction $\exists vTv \vee \exists vUv$ in one of these contexts is assumed rewritten $\exists v(Tv \vee Uv)$. For r.e. sets X, Y, we define

$$X \preccurlyeq Y\colon \{x\colon x \in X \preccurlyeq x \in Y\}, \quad X \prec Y\colon \{x\colon x \in X \prec x \in Y\},$$

where $x \in X$, $x \in Y$ abbreviate $\exists vTexv$ for appropriate e. Note that $X \preccurlyeq Y$ and $Y \prec X$ are simply the sets obtained by applying the Reduction Theorem to X, Y. (This notation is due to Dave Guaspari.)

1. A double dual completeness theorem. The main result of this note is the following

THEOREM. *Let (A, C), (B, D) be pairs of effectively inseparable r.e. sets with $A \subseteq B$, $C \subseteq D$. There is a recursive function f such that, for all x,*

$x \in A$ iff $fx \in A$ iff $fx \in B$;

$x \in C$ iff $fx \in C$ iff $fx \in D$.

In words, the conclusion of the theorem simply states that the pair (A, C) is uniformly many-one reducible to both pairs (A, C) and (B, D).

PROOF. The proof is simple but devious. By Smullyan's dual completeness

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result, there is a recursive function g such that, for all i, j, the function $[g(i,j)]$ reduces the pair $(W_i \preccurlyeq W_j, W_j \prec W_i)$ to (A, C). Apply Smullyan's Double Recursion Theorem (Rogers 1967, Theorem 11.10) to obtain indices a, c such that, for $f = [g(a, c)]$ and all x,

$$x \in W_a \Leftrightarrow [fx \in D \vee x \in A_. \preccurlyeq_. fx \in B \vee x \in C],$$
$$x \in W_c \Leftrightarrow [fx \in B \vee x \in C_. \prec_. fx \in D \vee x \in A].$$

Obviously, W_a and W_c are disjoint.

Claim 1. $W_a = A \preccurlyeq C = A$; $W_c = C \prec A = C$.

To see this, observe

$$\begin{aligned} x \in W_a &\Rightarrow x \in W_a - W_c \\ &\Rightarrow fx \in A \subseteq B \wedge fx \notin D, \quad \text{since } A \cap D = \varnothing \\ &\Rightarrow x \in A_. \preccurlyeq_. fx \in B \vee x \in C, \quad \text{by definition of } W_a \\ &\Rightarrow x \in A. \end{aligned}$$

Similarly, $x \in W_c \Rightarrow x \in C$. But also,

$$x \in A \Rightarrow x \in W_a \vee x \in W_c \Rightarrow x \in W_a,$$

since $x \in W_c$ yields $x \in C$ which is disjoint from A. Similarly $x \in C \Rightarrow x \in W_c$.

Claim 2. For all x,

$$x \in A \Leftrightarrow fx \in A, \quad x \in C \Leftrightarrow fx \in C.$$

This is trivial since $f = [g(a, c)]$ and $(A, C) = (W_a, W_c) = (W_a \preccurlyeq W_c, W_c \prec W_a)$.

Claim 3. For all x,

$$x \in A \Leftrightarrow fx \in B, \quad x \in C \Leftrightarrow fx \in D.$$

The left-to-right implications follow from Claim 2. For the other direction, assume first that $fx \in B$. A glance at the definition of W_a, W_c reveals that $x \in W_a$ or $x \in W_c$. The latter yields $fx \in D$, contrary to assumption. Thus $x \in W_a = A$. Similarly one shows $fx \in D$ implies $x \in C$. Q.E.D.

Obviously we can compose a reduction of (X, Y) to (A, C) with f to obtain a simultaneous reduction of any pair of disjoint r.e. sets to (A, C) and (B, D). A second corollary, noticed by J. R. Shoenfield, is this: For A, B, C, D as in the Theorem, any set X interpolated between A and B, $A \subseteq X \subseteq B$, has degree at least $\mathbf{0}'$. [N.B. Without C and D, this need not hold: Creative sets can have recursive interpolants.]

2. Some metamathematical applications. We give a few corollaries concerning the metamathematics of r.e. systems of arithmetic (for definiteness: extensions of Robinson's $\mathfrak{R}$) that were previously obtainable only via self-referential formulae (cf. Shepherdson 1960, Smoryński A).

DEFINITIONS. A formula $\varphi v_0 \cdots v_{n-1}$ *semirepresents* a relation $R \subseteq \omega^n$ in a theory $\mathfrak{T}$ iff, for all $x_0, \ldots, x_{n-1}$,

$$\mathfrak{T} \vdash \varphi\bar{x}_0 \cdots \bar{x}_{n-1} \Leftrightarrow Rx_0 \cdots x_{n-1}.$$

φ *dually semirepresents* a disjoint pair of relations, R, S iff φ, $\neg\varphi$ semirepresent R, S, respectively. φ *represents* R iff φ dually semirepresents R and its complement. A formula $\varphi v_0 \cdots v_n$ *semirepresents* (*represents*) a partial (total) function f iff (i) φ semirepresents (represents) the graph of f, and (ii) φ satisfies a unicity condition, say,

$$\mathfrak{T} \vdash \varphi v_0 \cdots v_{n-1} v \wedge \varphi v_0 \cdots v_{n-1} v' \rightarrow v = v'.$$

[This is stronger than necessary for most purposes.]

Corollary 1. *Let $\mathfrak{T}$ be a consistent r.e. extension of $\mathfrak{R}$. For any disjoint pair, R, S of n-ary r.e. relations, there is a formula $\varphi v_0 \cdots v_{n-1} \in \Sigma_1$ which dually semirepresents R, S in $\mathfrak{T}$; and, moreover, $\varphi v_0 \cdots v_{n-1}$ defines R in the set of natural numbers.*

Proof. Obviously we can assume the Theorem proven for n-ary relations. Moreover, by Smullyan's Dual Completeness Theorem, we can assume R, S to be effectively inseparable. So let ψ_0, ψ_1 be Σ_1 definitions of R, S and let $A = R$, $C = S$, $B = \{(x_0, \ldots, x_{n-1}): \mathfrak{T} \vdash (\psi_0 \preccurlyeq \psi_1)\bar{x}_0 \cdots \bar{x}_{n-1}\}$, and $D = \{(x_0, \ldots, x_{n-1}): \mathfrak{T} \vdash \neg(\psi_0 \preccurlyeq \psi_1)\bar{x}_0 \cdots \bar{x}_{n-1}\}$. Now simply define $\varphi v_0 \cdots v_{n-1}$:

$$\exists v'_0 \cdots v'_{n-1}\left[\chi v_0 \cdots v_{n-1} v'_0 \cdots v'_{n-1} \wedge (\psi_0 \preccurlyeq \psi_1) v'_0 \cdots v'_{n-1}\right],$$

where $\chi \in \Sigma_1$ represents the recursive function f of the Theorem. Q.E.D.

The correctness of the semirepresentation of R is the novel feature of this proof. While it comes free with Shepherdson's proof via self-referential formulae, the correctness has either been lacking in recursion-theoretic proofs of Corollary 1 (Ehrenfeucht and Feferman 1960, Putnam and Smullyan 1960), or has resulted in non-Σ_1 semirepresentations (Hájková and Hájek 1972).

Corollary 2. *The dual semirepresentation φ for disjoint R, S can be chosen uniformly in an r.e. sequence, $\mathfrak{T}_0, \mathfrak{T}_1, \ldots,$ of consistent extensions of $\mathfrak{R}$.*

The proof is as before: Let B_i, D_i be the sets of tuples provably in, respectively out of, $R \preccurlyeq S$ in $\mathfrak{T}_i$ and let $B = \bigcup_i B_i \preccurlyeq \bigcup_i D_i$, $D = \bigcup_i D_i \prec \bigcup_i B_i$.

Again, this result was originally quite easily proven by means of formal diagonalization.

Corollary 3. *Let f be partial recursive; $\mathfrak{T}_0, \mathfrak{T}_1, \ldots$ an r.e. sequence of consistent extensions of $\mathfrak{R}$. There is a formula $\varphi v_0 \cdots v_n \in \Sigma_1$ which correctly uniformly semirepresents f in each $\mathfrak{T}_i$. Moreover, we can assume*

$$\mathfrak{T}_i \vdash \neg\varphi\bar{x}_0 \cdots \bar{x}_{n-1}\bar{y} \Leftrightarrow \exists z \neq y(fx_0 \cdots x_{n-1} = z).$$

Again the result is sharper than the original recursion-theoretic result (Ritchie and Young 1968/1969). We omit the proof.

As a final application we have

Corollary 4. *Let $\mathfrak{T}_0 \subsetneq \mathfrak{T}_1$ be consistent r.e. extensions of $\mathfrak{R}$ and let*

$R_0 \subseteq R_1$ be n-ary r.e. relations. There is a formula φ such that φ semirepresents R_i in $\mathfrak{T}_i$.

PROOF. We shall cheat slightly. Di Paola 1966 shows that there is a ψ_0 which semirepresents R_0 in $\mathfrak{T}_0$ and ω^n in $\mathfrak{T}_1$. So let ψ_1 uniformly semirepresent R_1 in $\mathfrak{T}_0$, $\mathfrak{T}_1$ and define $\varphi = \psi_0 \wedge \psi_1$. Q.E.D.

Di Paola's full result required there to be a recursive interpolant between R_0 and R_1.

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