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A COMPLETE SPACE OF VECTOR-VALUED MEASURES

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ABSTRACT. Let X be a Hausdorff uniform space and E a Fréchet space (or more generally an LF-space) with dual F. Let $U^c(X, E)$ denote the uniformly continuous functions from X into E which have a precompact range, and let $U^c(X, E)$ have the topology of uniform convergence. Let $L(X, F)$ be the space of all F-valued measures on X with finite support, and let $L(X, F)$ be given the topology of uniform convergence over the uniformly equicontinuous subsets of $U^c(X, E)$ having a common precompact range in E. The main result in the paper is a characterization of the completion of $L(X, F)$ under this topology.

Introduction. Suppose X is a completely regular Hausdorff space. Let L be the space of all real-valued functions on X which are zero on the complement of a finite subset of X. X may be identified with a subset of L via the mapping $x \rightsquigarrow \chi_{\{x\}}$ where $\chi_{\{x\}}$ is the indicator function of the set $\{x\}$. There is a finest locally convex topology e on L whose restriction to X is the original topology on X. In **[4]** and **[5]**, Katětov began the study of the completion of (L, e). In a recent paper **[6]** one of the authors was able to characterize the completion of (L, e) and; in fact, showed that the completion of L for several natural locally convex topologies could be realized as certain spaces of Baire measures on X. More recently, Z. Frolík **[2]** and **[3]** has considered a similar situation with $(X, \mathfrak{U})$ being a Hausdorff uniform space. It is the purpose of this paper to extend the investigation of this completion problem in another direction by considering X to be a Hausdorff uniform space and by replacing the real number system by a general LF-space as the range of the functions in L. The characterization of the completion in this case is contained in Theorem 2.8 (which is the main result of the paper).

1. Preliminary results. Let (E, F) be a pairing of real linear spaces where $E \neq \{0\}$, $F \neq \{0\}$. Let F be *total* over E; that is, for each $0 \neq a \in E$ there is an $f \in F$ such that $(a, f) \neq 0$. Let E be given a locally convex topology $\mathfrak{T}_{\mathfrak{S}}$ which is the topology of uniform convergence on the members of some family $\mathfrak{S}$ of subsets of F. It will be assumed that $(E, \mathfrak{T}_{\mathfrak{S}})$ is complete. The space of all uniformly continuous functions from $(X, \mathfrak{U})$ into E which have a bounded range is denoted by $U^b((X, \mathfrak{U}), E)$.

Let $\mathfrak{B}_E$ be the collection of all closed, convex, balanced neighborhoods of zero in E. For each $V \in \mathfrak{B}_E$ and $f \in U^b((X, \mathfrak{U}), E)$, define $\|f\|_{V,\infty} =$

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$\sup\{p_V(f(x))|x \in X\}$ where p_V is the support functional determined by V. The family $\{\|\cdot\|_{V,\infty}|V \in \mathfrak{B}_E\}$ of seminorms generates a locally convex linear topology τ on $U^b((X, \mathfrak{U}), E)$ called the *topology of uniform convergence* on $U^b((X, \mathfrak{U}), E)$. The following may be proved in a straightforward manner.

(1) E Hausdorff $\Rightarrow (U^b((X, \mathfrak{U}), E), \tau)$ is Hausdorff.

(2) E complete $\Rightarrow (U^b((X, \mathfrak{U}), E), \tau)$ is complete.

(3) E metrizable $\Rightarrow (U^b((X, \mathfrak{U}), E), \tau)$ is metrizable.

(4) E normable $\Rightarrow (U^b((X, \mathfrak{U}), E), \tau)$ is normable.

Let $U^c((X, \mathfrak{U}), E)$ be the subspace of $U^b((X, \mathfrak{U}), E)$ consisting of those functions with a precompact range. It is not hard to show that $U^c((X, \mathfrak{U}), E)$ is a closed subspace of $U^b((X, \mathfrak{U}), E)$.

The space $L(X, F)$ of *molecular measures* from X into F is the space of all functions from X into F which are zero except on a finite subset of X. $L(X, F)$ is a real linear space. If f is any function from X into E and if $m \in L(X, F)$, define $\langle f, m\rangle = \Sigma\{(f(x), m(x))|x \in X\}$ where $(\cdot, \cdot)$ denotes the bilinear form pairing E and F. Then $\langle\cdot, \cdot\rangle$ is a bilinear form pairing each of the spaces of uniformly continuous functions described above with the space $L(X, F)$. For $B \subset U^b((X, \mathfrak{U}), E)$, B° and $B^{\circ\circ}$ will denote the polar and bipolar, respectively, relative to this pairing. If $Y \subset X$, then $B[Y] = \{f(y)|f \in B$ and $y \in Y\}$. (For simplicity, $B[x]$ will denote $B[\{x\}]$ for $x \in X$.)

If g is a linear functional on $L(X, F)$ and if $x \in X$, define a linear functional $f(x)$ on F by the equations:

$$(f(x), t) = \langle \chi_{\{x\}}(t), g\rangle \quad \text{for all } t \in F.$$

(Of course, $\chi_{\{x\}}$ is the indicator function for the set $\{x\}$.) The map $g \rightsquigarrow f$ is easily seen to be a one-to-one representation of the algebraic dual of $L(X, F)$ onto the space of all functions from X into the algebraic dual of F. This representation will be used throughout this work. The next two lemmas collect some basic facts which will be used below. The proofs are straightforward and will be omitted.

LEMMA 1.1. *Let $G \neq \phi$ be a subset of E and let $G^{\circ\circ}$ denote the bipolar of G for the pair (E, F).* (a) *If G is contained in a finite dimensional subspace of E so is $G^{\circ\circ}$.* (b) *If G is $\mathfrak{T}_\mathfrak{S}$-bounded, so is $G^{\circ\circ}$.* (c) *If G is $\mathfrak{T}_\mathfrak{S}$-precompact, so is $G^{\circ\circ}$.*

LEMMA 1.2. *Let $B \subset U^b((X, \mathfrak{U}), E)$ have the property that $B[x]$ is $\mathfrak{T}_\mathfrak{S}$-precompact for each $x \in X$ and let g be a linear functional on $L(X, F)$ which is bounded on B°. Then g: $X \to E$. In fact if M is an upper bound for $\{|\langle g, m\rangle| \,|m \in B^\circ\}$, then $g(x) \in M(B[x])^{\circ\circ}$ for all $x \in X$. In addition, if $|\langle g, m\rangle| \leqslant 1$ for all $m \in B^\circ$ then for any $x_1, x_2 \in X$ and any $t \in F$, it is true that $|(g(x_1) - g(x_2), t)| \leqslant \sup\{|(f(x_1) - f(x_2), t)| \,|f \in B\}$.*

A family $B \subset U^b((X, \mathfrak{U}), E)$ is called *uniformly equicontinuous* if for each $V \in \mathfrak{B}_E$, there is a $U \in \mathfrak{U}$ such that $f(x) - f(y) \in V$ for all $(x, y) \in U$

and all $f \in B$. Let $\mathcal{E}^c((X, \mathfrak{U}), E)$ denote the family of all sets $B \subset U^b((X, \mathfrak{U}), E)$ such that (1) B is uniformly equicontinuous and (2) $B[X]$ is a $\mathfrak{T}_{\mathfrak{S}}$-precompact subset of E. The following lemma is a result of Lemma 1.1.

LEMMA 1.3. (1) *If* $B \subset U^b((X, \mathfrak{U}), E)$ *then for all* $x \in X$, $B^{\circ\circ}[x] \subset (B[x])^{\circ\circ}$.

(2) *If* $B \in \mathcal{E}^c((X, \mathfrak{U}), E)$, *then* $B^{\circ\circ} \in \mathcal{E}^c((X, \mathfrak{U}), E)$.

Let u^c *be the topology on* $L(X, F)$ *or uniform convergence on the members of* $\mathcal{E}^c((X, \mathfrak{U}), E)$. *This topology is a locally convex, Hausdorff linear topology on* $L(X, F)$.

THEOREM 1.4. *The dual of* $(L(X, F), u^c)$ *is* $U^c((X, \mathfrak{U}), E)$.

PROOF. By the way in which u^c is defined, it is immediate that $U^c((X, \mathfrak{U}), E) \subset (L(X, F), u^c)'$.

Suppose $g \in (L(X, F), u^c)'$. Then there is a $B \in \mathcal{E}^c((X, \mathfrak{U}), E)$ such that $|\langle g, m\rangle| \leqslant 1$ for all $m \in B^\circ$. By Lemma 1.2, $g: X \to E$. Let $S \in \mathfrak{S}$ and $\varepsilon > 0$ be given. Then there is an entourage $W \in \mathfrak{U}$ such that $f(x) - f(y) \in \varepsilon S^\circ$ whenever $(x, y) \in W$ and $f \in B$. That is, for $t \in S$, $(x, y) \in W$, and $f \in B$; $|(f(x) - f(y), t)| \leqslant \varepsilon$. It follows that $|(g(x) - g(y), t)| \leqslant \varepsilon$ for all $(x, y) \in W$ and $t \in S$. Thus $g: X \to E$ is uniformly continuous. Also for every $x \in X$, $g(x) \in B[x]^{\circ\circ} \subset B[X]^{\circ\circ}$ which is precompact by Lemma 1.1 and the fact that $B[X]$ is precompact. Thus $g[X]$ is precompact. □

2. The completion. If $(X, \mathfrak{U})$ is a Hausdorff uniform space and if U^b is the space of all uniformly continuous, bounded, real-valued functions on X, let M be the dual of U^b provided with the topology of uniform convergence on the uniformly bounded and uniformly equicontinuous subsets of U^b. The closure of L in M is called the space of *uniform measures*. (In **[2]**, Frolík characterizes these measures.)

Let m be a linear functional on $U^b((X, \mathfrak{U}), E)$. For each $t \in E$, let m_t be the linear functional defined on $U^b((X, \mathfrak{U}), \mathbf{R})$ by $(\phi, m_t) = \langle \phi t, m\rangle$ for all $\phi \in U^b((X, \mathfrak{U}), \mathbf{R})$.

PROPOSITION 2.1. *If* m *is in the completion of* $(L(X, F), u^c)$ *then* m_t *is a uniform measure for all* $t \in E$ *and* m *is a linear functional on* $U^c((X, \mathfrak{U}), E)$ *which is sequentially continuous for the topology* τ *of uniform convergence.*

PROOF. Let $m \in (L(X, F), u^c)^\wedge$ (the completion of $(L(X, F), u^c)$). By the completion theorem of Grothendieck **[7**, p. 270**]** and Theorem 1.4, m is a linear functional on $U^c((X, \mathfrak{U}), E)$.

Now let $t \in E$ be fixed and let $\{\phi_i | i \in I\}$ be a net in $U^b((X, \mathfrak{U}), \mathbf{R})$ which is uniformly bounded and uniformly equicontinuous and which satisfies $\phi_i \to 0$ pointwise on X. Then $\{\phi_i t | i \in I\} \in \mathcal{E}^c((X, \mathfrak{U}), E)$ and $\phi_i t \to 0$ in the $\sigma(U^c((X, \mathfrak{U}), E), L(X, F))$-sense. Thus $\langle \phi_i t, m\rangle \to 0$, and so $(\phi_i, m_t) \to 0$. It follows that m_t is a uniform measure for all $t \in E$.

Now let $\{f_n | n \in \mathbf{N}\}$ be a sequence in $U^c((X, \mathfrak{U}), E)$ which converges uniformly to zero. In view of the completion theorem of Grothendieck it will be established that $\langle f_i, m\rangle \to 0$ if it can be shown that $\{f_n | n \in \mathbf{N}\} \in \mathcal{E}^c((X, \mathfrak{U}), E)$ and $f_n \to 0$ in the $\sigma(U^c((X, \mathfrak{U}), E), L(X, F))$-sense. All the necessary properties are clear except the fact that $G = \bigcup\{f_n[X] | n \in \mathbf{N}\}$ is precompact. This property is now shown.

Let U be a closed, convex, balanced neighborhood of zero in E. Let $n_0 \in \mathbf{N}$ be such that $f_n[X] \subset U$ for $n \geqslant n_0$. Since $\bigcup\{f_n[X] | n < n_0\}$ is a finite union of precompact sets, it is precompact. Thus there is a finite set $\{t_1, t_2, \ldots, t_k\}$ of points in G such that $\bigcup\{f_n[X] | n < n_0\} \subset \bigcup\{t_j + U | j = 1, \ldots, k\}$. Let $t_{k+1} = 0$. Then $G \cup \{0\} \subset \bigcup\{t_j + U | j = 1, 2, \ldots, k + 1\}$. It follows that $G \cup \{0\}$, and hence G, is precompact.

The above yields an upper bound for $(L(X, F), u^c)\hat{}$. Before giving a lower bound, it is helpful to give a useful lemma (Lemma 2.2) whose proof may be found in **[8]**. Recall that if (A, p) is a seminormed space, a subset H of A is said to have a *nearest point map* π if π is a function from A into H such that for all $a \in A$ and all $h \in H - \{\pi(a)\}$, $p(a - \pi(a)) < p(a - h)$.

LEMMA 2.2. *Let (A, p) be a seminormed space. Let $N_p = \{a \in A | p(a) = 0\}$ and let A_p be a linear complement of N_p. If A_p with the subspace topology is separable, there is a seminorm q on A which is equivalent to p and such that every nonempty, compact, convex subset of A_p has a continuous nearest point map for the seminorm q.*

PROPOSITION 2.3. *If m is a uniformly continuous linear functional on $U^c((X, \mathfrak{U}), E)$ such that m_t is a uniform measure for every $t \in E$, then for every net $\{f_i | i \in I\} \in \mathcal{E}^c((X, \mathfrak{U}), E)$ such that $f_i \to 0$ in the $\sigma(U^c((X, \mathfrak{U}), E), L(X, F))$-sense it is true that $\langle f_i, m\rangle \to 0$.*

PROOF. Let U be a closed, convex, balanced neighborhood of zero in E such that $|\langle f, m\rangle| \leqslant 1$ for all f such that $f(x) \in U$ for all $x \in X$. Let $N_U = \{s \in E | p_U(s) = 0\}$ and let E_U be a linear complement of N_U. For any $t \in E$, let $t_N + t_U$ be the unique representation of t as a sum where $t_N \in N_U$ and $t_U \in E_U$. Clearly p_U restricted to E_U is a norm. Also, since $p_U(t) = p_U(t_U)$ it follows that the map $d\colon (E, \mathfrak{T}_{\mathbb{S}}) \to (E_U, p_U)$ by $t \to t_U$ is continuous.

Since $\{f_i | i \in I\} \in \mathcal{E}^c((X, \mathfrak{U}), E)$ there is a compact subset G of E such that $f_i[X] \subset G$ for all $i \in I$. It may be assumed by Lemma 1.1 that $G = G^{\circ\circ}$. Let $G_d = d[G]$. Then G_d is a compact subset of (E_U, p_U). Let D_U be the smallest subspace of (E_U, p_U) which contains G_d. Since G_d is compact, (D_U, p_U) is separable. Let $A = N_U + D_U$ be equipped with the seminorm p_U. There is by Lemma 2.2 a seminorm q on A which is equivalent to p_U such that every nonempty compact, convex subset of D_U has a nearest point map. Let $\alpha, \beta > 0$ be constants such that $p_U(a) \leqslant \alpha q(a)$ and $q(a) \leqslant \beta p_U(a)$ for all $a \in A$.

Now let $\varepsilon > 0$ be given. Let $\{t_1, t_2, \ldots, t_n\}$ be a finite ε-net of G_d for the norm p_U on D_U. Then $G \subset \{t_k + \varepsilon U | k = 1, 2, \ldots, n\}$. Let H be the closed, absolutely convex hull of $\{t_1, t_2, \ldots, t_n\}$ in (D_U, p_U). Then H is a compact (closed and bounded) subset of a finite dimensional Hausdorff space K in

D_U. By Lemma 2.2 there is a continuous map π: $(A, p_U) \to H$ such that $q(a - \pi(a)) < q(a - h)$ for all $a \in A$ and all $h \in H - \{\pi(a)\}$. Since the topology determined by p_U on A is weaker than the restriction of $\mathfrak{T}_{\mathbb{S}}$ to A, and since these topologies agree on K, the nearest point map π: $A \to H$ is continuous when A and H are provided with the restriction of $\mathfrak{T}_{\mathbb{S}}$.

The following facts are now established:

(1) $\{\pi \circ f_i | i \in I\} \in \mathcal{E}^c((X, \mathfrak{U}), K)$.

(2) $\pi \circ f_i \to 0$ in the $\sigma(U^c((X, \mathfrak{U}), E), L(X, F))$-sense.

(3) $p_U(f_i(x) - \pi \circ f_i(x)) \leqslant \alpha\beta\varepsilon$ for all $x \in X$ and all $i \in I$.

In order to verify (1), observe that since π is continuous, $\pi[G]$ is a compact subset of K. Also $\pi \circ f_i[X] \subset \pi[G]$ for all $i \in I$. Since G is compact, π is uniformly continuous on G. It follows that $\{\pi \circ f_i | i \in I\}$ is a uniformly equicontinuous family. Thus (1) is established.

In order to verify (2), observe that since G is compact and since $\sigma(E, F)$ is weaker than $\mathfrak{T}_{\mathbb{S}}$, the topologies agree on G. Thus $f_i(x) \to 0$ in the $\mathfrak{T}_{\mathbb{S}}$-sense for all $x \in X$. In particular $p_U(f_i(x)) \to 0$ for all $x \in X$ and so $q(f_i(x)) \to 0$ for all $x \in X$. Since $0 \in H$, it is true that $q(f_i(x) - \pi \circ f_i(x)) \leqslant q(f_i(x))$. Thus

$$q(\pi \circ f_i(x)) \leqslant q(\pi \circ f_i(x) - f_i(x)) + q(f_i(x)) \leqslant 2q(f_i(x)) \to 0.$$

Since the topology determined by q and $\sigma(E, F)$ agree on K, it follows that $\pi \circ f_i(x) \to 0$ in the $\sigma(E, F)$-sense for all $x \in X$. This establishes (2).

Finally let $x \in X$ and $i \in I$. Then $f_i(x) \in G \Rightarrow p_U(f_i(x) - t_k) < \varepsilon$ for some $k \in \{1, 2, \ldots, n\}$. Thus $q(f_i(x) - t_k) < \alpha\varepsilon$ so $q(f_i(x) - \pi \circ f_i(x)) < \alpha\varepsilon$. It follows that $p_U(f_i(x) - \pi \circ f_i(x)) < \alpha\beta\varepsilon$. Thus (3) is established.

Now let $\{s_1, s_2, \ldots, s_p\}$ be a basis for K. Then by (1), there is a uniformly bounded and uniformly equicontinuous family $\{\phi_{ij} | i \in I; j = 1, 2, \ldots, p\}$ of real valued functions on X such that $\pi \circ f_i(x) = \sum_{j=1}^{p} \phi_{ij}(x) s_j$ for all $i \in I$. It follows from (2) that $\phi_{ij}(x) \to 0$ for each $x \in X$. Since m_{s_j} is a uniform measure for each $j \in \{1, 2, \ldots, p\}$, it follows that

$$\begin{aligned} |\langle f_i, m \rangle| &\leqslant |\langle f_i - \pi \circ f_i, m \rangle| + |\langle \pi \circ f_i, m \rangle| \\ &\leqslant |\langle f_i - \pi \circ f_i, m \rangle| + \sum_{j=1}^{p} |(\phi_{ij}, m_{s_j})| \leqslant \alpha\beta\varepsilon + \sum_{j=1}^{p} |(\phi_{ij}, m_{s_j})|. \end{aligned}$$

Thus lim sup $|\langle f_i, m \rangle| \leqslant \alpha\beta\varepsilon$. Since the choice of $\varepsilon > 0$ was arbitrary, $\langle f_i, m \rangle \to 0$. □

The following corollary is immediate from Proposition 2.3 and Grothendieck's completion theorem.

COROLLARY 2.4. *If m is a linear functional on $U^c((X, \mathfrak{U}), E)$ which is continuous for the topology τ of uniform convergence on $U^c((X, \mathfrak{U}), E)$ and if m_t is a uniform measure for all $t \in E$, then $m \in (L(X, F), u^c)\hat{}$.*

The above corollary gives a lower bound for $(L(X, F), u^c)\hat{}$. Combining this with Proposition 2.1, yields the following theorem.

THEOREM 2.5. *If E is a Fréchet space, the completion of $(L(X, F), u^c)$ is the* *space of all linear functionals m on* $U^c((X, \mathfrak{U}), E)$ *which are continuous for the topology of uniform convergence on* $U^c((X, \mathfrak{U}), E)$ *and for which* m_t *is a uniform measure for all* $t \in E$.

A Hausdorff topological vector space A is called an *LF-space* if it is the strict inductive limit of a sequence of Fréchet spaces. That is, A is an LF-space if there is a sequence $\{A_m | n \in \mathbf{N}\}$ of subvector spaces of A such that for each $n \in \mathbf{N}$; $A_n \subset A_{n+1}$; A_n has a topology $\mathcal{G}_n$ making $(A_n, \mathcal{G}_n)$ a Fréchet space; $\mathcal{G}_{n+1}$ restricted to A_n is $\mathcal{G}_n$; $A = \bigcup\{A_n | n \in \mathbf{N}\}$; and the topology of A is the finest locally convex topology on A making each of the injections $i_n\colon A_n \hookrightarrow A$ continuous. The sequence $\{(A_n, \mathcal{G}_n) | n \in \mathbf{N}\}$ is called a *sequence of definition* for the LF-space A. The following two results are used in the proof of Theorem 2.8. The proofs of these propositions may be found in [**9**, 6.4 and 6.5 respectively].

PROPOSITION 2.6. *If* $\{(A_n, \mathcal{G}_n) | n \in \mathbf{N}\}$ *is an increasing sequence of locally convex Hausdorff topological vector spaces such that* $\mathcal{G}_{n+1}$ *induces* $\mathcal{G}_n$ *for all n and if the vector space A is the union of the subspaces* A_n, *then the inductive topology on A with respect to the imbeddings* $A_n \hookrightarrow A$ *is Hausdorff and induces* $\mathcal{G}_n$ *and* A_n *for all* $n \in \mathbf{N}$.

PROPOSITION 2.7. *Let* $(A, \mathcal{G})$ *be the inductive limit of the sequence* $\{(A_n, \mathcal{G}_n) | n \in \mathbf{N}\}$ *where* A_n *is closed in* $(A_{n+1}, \mathcal{G}_{n+1})$ *for all* $n \in \mathbf{N}$. *A subset* $B \subset A$ *is bounded in* $(A, \mathcal{G})$ *if and only if for some* $n \in \mathbf{N}$, *B is a bounded subset of* $(A_n, \mathcal{G}_n)$.

THEOREM 2.8. *If* $(E, \mathfrak{T}_{\mathbb{S}})$ *is an LF-space, then the completion of* $(L(X, F), u^c)$ *is the space of all linear functionals m on* $U^c((X, \mathfrak{U}), E)$ *which are sequentially continuous for the topology* τ *of uniform convergence on* $U^c((X, \mathfrak{U}), E)$ *and for which* m_t *is a uniform measure for all* $t \in E$.

PROOF. In view of Proposition 2.1, the proof will be complete if it can be shown that every τ-sequentially continuous linear functional m on $U^c((X, \mathfrak{U}), E)$ for which m_t is a uniform measure for all $t \in E$ is in $(L(X, F), u^c)^\wedge$. Let m denote such a functional. Let $\{f_i | i \in I\}$ be a net such that $\{f_i | i \in I\} \in \mathcal{E}^c((X, \mathfrak{U}), E)$ and such that $f_i \to 0$ on the $\sigma(U^c((X, \mathfrak{U}), E), L(X, F))$-sense. It is shown below that $\langle f_i, m\rangle \to 0$. This establishes the theorem.

Let $\{(E_n, \mathfrak{T}_n) | n \in \mathbf{N}\}$ be a sequence of definition for $(E, \mathfrak{T}_{\mathbb{S}})$. By Proposition 2.6, $\mathfrak{T}_{\mathbb{S}}$ restricted to E_n is $\mathfrak{T}_n$. It follows that $(E_n, \mathfrak{T}_n)$ satisfies the assumptions placed on E in this paper. Now for each $n \in \mathbf{N}$, $U^c((X, \mathfrak{U}), E_n) \subset U^c((X, \mathfrak{U}), E)$. Further, since $(E_n, \mathfrak{T}_n)$ is a Fréchet space, $U^c((X, \mathfrak{U}), E_n)$ provided with the topology τ_n of uniform convergence on $U^c((X, \mathfrak{U}), E_n)$ is a Fréchet space. Also the restriction of τ to $U^c((X, \mathfrak{U}), E_n)$ is weaker than τ_n.

Since $m\colon U^c((X, \mathfrak{U}), E) \to \mathbf{R}$ is τ-sequentially continuous, its restriction to $U^c((X, \mathfrak{U}), E_n)$ is τ_n-sequentially continuous and thus τ_n-continuous. Thus for each $n \in \mathbf{N}$, the restriction of m to $U^c((X, \mathfrak{U}), E_n)$ is a τ_n-continuous linear functional with m_t a uniform measure for all $t \in E_n$.

Now $\{f_i | i \in I\} \in \mathcal{E}^c((X, \mathfrak{U}), E_n)$, so $\bigcup\{f_i[X] | i \in I\}$ is a bounded subset of E. By Proposition 2.7, $\{f_i[X] | i \in I\} \subset E_{n_0}$ for some $n_0 \in \mathbf{N}$. That $\{f_i | i \in I\} \in \mathcal{E}^c((X, \mathfrak{U}), E_{n_0})$ follows easily. Also, $f_i \to 0$ in the $\sigma(U^c((X, \mathfrak{U}), E_{n_0}), L(X, F))$-sense. Applying Proposition 2.3, $\langle f_i, m\rangle \to 0$.

It should be noted that results for cases where X is considered as a completely regular Hausdorff space are immediate from the above if one considers the finest uniformity $\mathfrak{U}$ compatible with the topology. Also, there are several other natural topologies which may be placed on $L(X, F)$. These will be the subject of future notes.

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