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# NORMAL AND QUASINORMAL COMPOSITION OPERATORS

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ABSTRACT. A bounded linear operator $C_T$ on $L^2(X, \Sigma, m)$ is a composition operator if it is induced by a point mapping $T: X \to X$ via $C_T f = f \circ T$.

Normal and quasinormal composition operators on a finite measure space are characterized: $C_T$ is normal iff $T$ is measure preserving and $T^{-1}(\Sigma)$ is (essentially) all of $\Sigma$; $C_T$ is quasinormal iff $T$ is measure preserving.

In what follows, the composition operator $C_T$ on $L^2(X, \Sigma, m)$, a sigma-finite measure space, will always be a bounded linear operator; in terms of $T$ this means that $T$ is measurable, $m \circ T^{-1}$ is absolutely continuous with respect to $m$, and the Radon-Nikodým derivative $h = dm \circ T^{-1}/dm$ is essentially bounded. Conversely, composition with any $T$ satisfying these conditions induces a bounded linear operator $C_T$ on $L^2(X, \Sigma, m)$ [**1**, pp. 663–665] and [**3**].

The change of variable equation

$$\int C_T f \, dm = \int f \circ T \, dm = \int fh \, dm \qquad (1)$$

is basic.

The following concept arises in understanding normal composition operators:

DEFINITION 1. Let $(X, \Sigma, m)$ be a measure space and $T: X \to X$ a measurable function on $X$. The sigma algebra $T^{-1}(\Sigma)$ *is essentially all of* $\Sigma$ iff given $A$ in $\Sigma$ there is a $B$ in $\Sigma$ with the symmetric difference $T^{-1}(B) \triangle A = (T^{-1}(B) - A) \cup (A - T^{-1}(B))$ having $m(T^{-1}(B) \triangle A) = 0$.

Since the elements of $L^2$ are equivalence classes of functions under the equivalence relation "equal almost everywhere", changing $T$ on a set of measure zero does not change $C_T$. Consequently, any characterization of $C_T$ in terms of $T$ must use "essential" properties of $T$ as in Definition 1, invariant under the class of maps inducing the same $C_T$.

LEMMA 1. *Let $(X, \Sigma, m)$ be a sigma-finite measure space. The composition operator $C_T$ has dense range iff $T^{-1}(\Sigma)$ is essentially all of $\Sigma$.*

PROOF. Suppose $C_T$ has dense range and let $A$ be a set in $\Sigma$ of finite measure. By hypothesis there are functions $f_n$ in $L^2$ with $C_T f_n \to \chi_A$, by passing to a subsequence $C_T f_n \to \chi_A$ a.e. Since each function $C_T f_n$ is

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measurable with respect to the sigma-algebra $T^{-1}(\Sigma)$, the equivalence class $\chi_A$ is also $T^{-1}(\Sigma)$ measurable. Which is to say that there is a $B$ in $\Sigma$ with $\chi_{T^{-1}(B)} = \chi_A$ a.e., so $m(T^{-1}(B) \triangle A) = 0$. Since this is true for each set of finite measure, it is true for each measurable set in the sigma-finite measure space.

Suppose that $T^{-1}(\Sigma)$ is essentially all of $\Sigma$ and let $A$ be a set of finite measure. There is a set $B$ in $\Sigma$ with $m(T^{-1}(B) \triangle A) = 0$ by hypothesis. Using sigma-finiteness write $B$ as the union of an increasing sequence $\{B_n\}$ of sets of finite measure. Then $C_T \chi_{B_n}$ increases monotonically to $\chi_A$ a.e., and so $\|C_T \chi_{B_n} - \chi_A\| \to 0$. Thus the closure of the range of $C_T$ contains all simple functions and is therefore all of $L^2$. Q.E.D.

Singh shows in [7] that if $T$ is one-to-one, with a measurable $S$ on $X$ satisfying $STx = x$ a.e., then $C_T$ has dense range. This is immediate from Lemma 1.

LEMMA 2. *Let $(X, \Sigma, m)$ be sigma-finite. The composition operator $C_T$ is normal iff*

(a) $T^{-1}(\Sigma)$ *is essentially all of* $\Sigma$, *and*

(b) $h \circ T = h$ *a.e.* (*Recall* $h = dm \circ T^{-1}/dm$.)

PROOF. For any $f$ and $g$ in $L^2$,

$$(C_T^* C_T f, g) = (C_T f, C_T g) = \int (f \circ T)(\bar{g} \circ T)\, dm = \int f\bar{g}h\, dm,$$

so $C_T^* C_T f = hf$. And, if $f$ belongs to the range of $C_T$, $f = C_T f_0$, then $C_T C_T^* f = C_T C_T^* C_T f_0 = C_T(hf_0) = (h \circ T)f$ [6].

Suppose $C_T$ is normal and write $X = \bigcup X_n$, a union of sets of finite measure. The function $\chi_{T^{-1}(X_n)}$ belongs to the range of $C_T$ and thus, from the above, $h = h \circ T$ a.e. on $T^{-1}(X_n)$ and so $h = h \circ T$ a.e. on $\bigcup T^{-1}(X_n) = X$. Let $A_n = \{x: h(x) \geqslant 1/n\}$. For $f$ in $L^2$, $g = (\chi_{A_n}/h)f$ also belongs to $L^2$ and $C_T C_T^* g = C_T^* C_T g = \chi_{A_n} f$. Because $\bigcup A_n = \sigma = \sigma(h) = \{x: h(x) > 0\}$, $\|\chi_{A_n} f - \chi_\sigma f\| \to 0$ and $C_T$ has range dense in $L^2(\sigma(h))$. Write $X = T^{-1}(\sigma) \cup T^{-1}(\sigma^c)$ and note that $m(T^{-1}(\sigma^c)) = 0$ from (1), since $\|C_T f\|^2 = \int |f|^2 h\, dm$. By (b), $T^{-1}(\sigma) = \sigma$ so $\sigma = X$ to within a set of measure zero. Hence $C_T$ has range dense in $L^2(X)$ and (a) follows from Lemma 1.

Suppose that (a) and (b) hold. As above, $C_T^* C_T f = hf = h \circ Tf = C_T C_T^* f$ for $f$ in the range of $C_T$. Hence $C_T^* C_T - C_T C_T^*$ is zero on the range of $C_T$ and so identically zero by (a) and Lemma 1, i.e. $C_T$ is normal. Q.E.D.

As it stands, Lemma 2 is a characterization of normal $C_T$, but it is not entirely satisfactory as it does not explicitly indicate the properties of $T$ involved. The answer to exactly which $T$ gives rise to a normal $C_T$ is hidden precisely in those solutions to the curious equation

$$h = h \circ T \tag{2}$$

relating $T$ and its Radon-Nikodým derivative $h = dm \circ T^{-1}/dm$.

I can determine these solutions for finite measure spaces; the surprising answer is that $h = 1$ a.e.

THEOREM 1. *Let* $(X, \Sigma, m)$ *be a finite measure space. The following are equivalent*:

1. $C_T$ *is normal,*
2. $C_T$ *is unitary,*
3. (a) $T^{-1}(\Sigma)$ *is essentially all of* $\Sigma$, *and*
(b) $T$ *is measure preserving.*

PROOF. Suppose that $C_T$ is normal. By Lemma 2, $h = h \circ T$, so $B = \{x: h(x) \geqslant 1\} = \{x: h(Tx) \geqslant 1\} = T^{-1}(B)$. Then

$$\|C_T\chi_B\|^2 = \int \chi_B h \, dm = \int (\chi_B \circ T)(h \circ T) \, dm = \int \chi_B h^2 \, dm.$$

Since $h^2 \geqslant h$ on $B$, $h^2 = h$ on $B$ and it follows that $h = 1$ on $B$ (a.e.). A similar argument applied to the set $C = \{x: h(x) < 1\}$ shows that $h = 0$ (a.e.) on $C$. Hence $h = \chi_B$. The equation $m(X) = m(T^{-1}(X)) = \int \chi_B \, dm = m(B)$ shows that $m(B^c) = 0$ and $h = 1$ a.e. Another use of equation (1) shows that $m(T^{-1}(A)) = \int \chi_A \, dm = m(A)$ and 3 (b) holds. Also, $C_T^* C_T f = C_T(C_T^* f) = hf = f$, $C_T$ is onto and Lemma 1 implies 3 (a).

Suppose that 3 (a) and 3 (b) hold. Because $T$ is measure preserving, $m(T^{-1}(A)) = m(A) = \int \chi_A \, dm$ for all measurable $A$. From the uniqueness of the Radon-Nikodým derivative $h$, $h = 1$ a.e. and thus $\|C_T f\|^2 = \int |f|^2 h \, dm = \|f\|^2$. The range of $C_T$ is then closed, and therefore all of $L^2(X)$ by Lemma 1. Each $f$ in $L^2$ belongs to the range of $C_T$, so as in the proof of Lemma 2, $C_T C_T^* f = C_T^* C_T f = hf = f$, and $C_T$ is unitary. Q.E.D.

In the case where $(X, \Sigma, m)$ is sigma-finite, simple examples show that $C_T$ can be normal without $T$ being measure preserving. (Consider the real line with Lebesgue measure and $Tx = 2x$.)

Similar behavior is possible for a generalized composition operator: In [4] Ridge defines a composition operator $C_T$ induced by a map $T: X_1 \to X$ on a measurable *subset* $X_1$ of $X$ by $C_T f = (f \circ T)\chi_{X_1}$. With this definition and $X$ of finite measure, for $C_T$ to be normal $C_T C_T^* 1 = [(C_T^* 1) \circ T]\chi_{X_1} = C_T^* C_T 1 = h$, and $h$ must vanish off $X_1$; thus $m(T^{-1}(X_1^c)) = \int \chi_{X_1^c} h \, dm = 0$ and $T$ essentially maps $X_1$ into $X_1$, which is basically the case considered here. However it is possible for such a generalized $C_T$ to be quasinormal without $T$ being measure preserving. (Consider $X = (0, 2)$ with Lebesgue measure, $X_1 = (0, 1)$, and $Tx = 2x$.)

In either case, $(X, \Sigma, m)$ sigma-finite or for Ridge's generalized composition operators, it would be interesting to characterize normal and quasinormal $C_T$.

Normal composition operators have now been characterized in terms of two properties of $T$. One of these properties was shown to correspond to a standard operator theoretic property of $C_T$ in Lemma 1. There is a correspondence for the other property, as is now shown.

LEMMA 3. *Let* $(X, \Sigma, m)$ *be a sigma-finite measure space. Then* $C_T$ *is quasinormal iff* $h = h \circ T$.

PROOF. By definition [2], $C_T$ is quasinormal iff $C_T$ commutes with $C_T^*C_T$. From $C_T^*C_Tf = hf$, $C_TC_T^*C_Tf = (h \circ T)(f \circ T)$ and $C_T^*C_TC_Tf = h(f \circ T)$. As in Lemma 2, $h = h \circ T$.

The converse follows from the calculation above. Q.E.D.

THEOREM 2. *Let $(X, \Sigma, m)$ be a finite measure space. The following are equivalent*:

1. *$C_T$ is quasinormal,*
2. *$C_T$ is an isometry,*
3. *$T$ is measure preserving.*

PROOF. Suppose $C_T$ is quasinormal. Arguing as in Theorem 1 shows that $T$ is measure preserving and so that $h = 1$ a.e. Thus

$$\|C_Tf\|^2 = \int |f|^2 h \, dm = \int |f|^2 \, dm, \tag{3}$$

and $C_T$ is an isometry.

Suppose $C_T$ is an isometry. (3) shows that $h = 1$ a.e. and thus that $T$ is measure preserving.

Suppose $T$ is measure preserving. Then $h = 1$ a.e. and Lemma 3 shows that $C_T$ is quasinormal. Q.E.D.

In [7] the result is given that for $T$ one-to-one, with $S$ measurable and $STx = x$ a.e., $C_T$ is normal iff $C_T$ is quasinormal. This follows as condition (a) of Lemma 2 is automatically satisfied for $T$ one-to-one.

EXAMPLE. A composition operator $C_T$ on $L^2(0, 1)$ which is quasinormal but not normal.

Consider the $T$ on $(0, 1)$ given by:

$$Tx = \begin{cases} 2x, & 0 \leqslant x \leqslant 1/2, \\ 2 - 2x, & 1/2 \leqslant x \leqslant 1. \end{cases}$$

For this $T$, $mT^{-1}(0, x) = m(0, x/2) + m(1 - x/2, 1) = m(0, x)$. It follows that $mT^{-1}(a, b) = m(a, b)$ and thus $m \circ T^{-1} \ll m$. Since each measurable set is nearly a disjoint union of intervals [5, pp. 62 and 71] $T$ is measure preserving and so $h = 1$. Alternately, a change of variable shows that $\|C_Tf\|^2 = \|f\|^2$ and $C_T$ is an isometry and thus $T$ measure preserving.

Since $h = 1$, $C_T$ is quasinormal by Lemma 3. Alternately, a direct computation shows that

$$C_T^*f(x) = (1/2)f(x/2) + (1/2)f(1 - x/2).$$

Thus $C_T$ commutes with the identity $C_T^*C_T$.

You can directly see that $C_T$ is not normal by considering the action of $C_TC_T^*$ and $C_T^*C_T$ on $\chi_{(0,1/2)}$. Alternately, for $g = \chi_{(0,1/2)} - \chi_{(1/2,1)}$,

$$\int \chi_{T^{-1}(A)} g \, dm = 0$$

for all measurable $A$. It follows, for example, that $(0, 1/2)$ does not essentially belong to $T^{-1}(\Sigma)$, $\Sigma$ the Lebesgue measurable sets.

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