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ON A DISCRETENESS CONDITION OF THE SPECTRUM OF SCHRÖDINGER OPERATORS WITH UNBOUNDED POTENTIAL FROM BELOW[1]

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ABSTRACT. We obtain a discreteness condition for the spectrum of the Schrödinger operator $-\Delta + V(x)$ in a case in which V is not bounded from below.

0. Introduction. Discreteness conditions of the spectrum of the Schrödinger operators $H = -\Delta + V(x)$ (Δ the Laplacian and V a potential) have been studied by several authors under the assumption that V is bounded from below (cf., e.g. A. M. Molchanov [5], M. Schechter [6], I. M. Glazman [4]). In a recent paper [1] we have proved that if $V \in L^2_{\text{loc}}(R^n)$, $V(x) > 0$ and $\int_{S(x_0)} V(x)^{-1}\,dx \to 0$ for $|x_0| \to +\infty$ ($S(x_0)$ is the unit sphere centered at x_0), then the spectrum $\sigma(h)$ of the selfadjoint realization h of H in $L^2(R^n)$ consists of a denumerable set of eigenvalues of finite multiplicity. The proof is based on a compact embedding theorem of M. Berger and M. Schechter [3] and on the use of a suitable class of weighted Sobolev spaces introduced by the authors [2]. In the present paper we generalize the result of [1] to a case in which V is not bounded from below; precisely, we assume that $V = V_1 + V_2$ where V_1, V_2 satisfy the following assumptions:

$$\exists k > 0 \text{ s.t. inf ess } V_1(x) > -k \text{ and } \int_{S(x_0)} dx/\left(V_1(x) + k\right) \to 0 \quad \text{for } |x_0| \to +\infty, \tag{0.1}$$

$$V_2 \in L^{n/2}(R^n) \quad \text{for } n \geqslant 2. \tag{0.2}$$

1. Some preliminaries. Let ρ_0, ρ_1 be two positive Lebesgue measurable functions on R^n, $p \in]1, +\infty[$. We denote by $\Gamma^{1,p}(R^n, \rho_0, \rho_1)$ the space of distribution u on R^n such that $(\rho_0)^{1/p} u \in L^p(R^n)$ and $(\rho_1)^{1/p}|\text{grad}\, u| \in L^p(R^n)$ equipped with the norm

$$\|u\|_{\Gamma^{1,p}(\mathbf{R}^n,\rho_0,\rho_1)} = \left(\int_{R^n} \rho_0(x)|u(x)|^p\,dx + \int_{R^n} \rho_1(x)|\text{grad}\, u(x)|^p\,dx\right)^{1/p}.$$

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$\mathring{\Gamma}^{1,p}(R^n, \rho_0, \rho_1)$ is the closure of $C_0^\infty(R^n)$ in $\Gamma^{1,p}(R^n, \rho_0, \rho_1)$. As usual, we put $\Gamma^{1,2}(R^n, 1, 1) = H^1(R^n)$. We shall denote by $S(x_0)$ the unit ball in R^n centered at $x_0 \in R^n$. Let us recall the following compact embedding theorem (cf. [**1**, Theorem 2.1]).

THEOREM 1.1. *Let us suppose that* $\inf \operatorname{ess} \rho_0(x) > 0$, $\inf \operatorname{ess} \rho_1(x) > 0$ *and* $\int_{S(x_0)} \rho_0(x)^{-1}\, dx \to 0$ *for* $|x_0| \to +\infty$. *Then the embedding* $\Gamma^{1,p}(R^n, \rho_0, \rho_1) \hookrightarrow L^p(R^n)$ *is compact.*

2. The results. Let $V \in L^1_{\text{loc}}(R^n)$ be a real potential which admits the decomposition $V = V_1 + V_2$, with V_1, V_2 measurable, and real functions satisfying the following assumptions:

(i) $\exists k > 0$ such that $\inf \operatorname{ess} V_1(x) > -k$,

(ii) $\int_{S(x_0)} (V_1(x) + k)^{-1}\, dx + k \to 0$ for $|x_0| \to +\infty$,

(iii) $V_2 \in L^{n/2}(R^n)$ for $n \geqslant 2$.

Let us consider the Hamiltonian operator formally defined by

$$Hu = -\Delta u + V(x)u$$

and the associated sesquilinear form

$$a(u, v) = \int \left\{ \sum_i u_{x_i}(x)\overline{v_{x_i}(x)} + V(x)u(x)\overline{v(x)} \right\} dx, \quad u, v \in C_0^\infty(R^n).$$

In the following we shall put $W = \mathring{\Gamma}^{1,2}(R^n, V_1 + k, 1)$ and its scalar product will be denoted by $(\cdot|\cdot)_W$.

PROPOSITION 2.1. *The form* $(a\,\cdot, \cdot)$ *can be continuously extended to* $W \times W$.

PROOF. By virtue of (iii) and the Sobolev embedding theorem, we have $\forall_v^u \in C_0^\infty(R^n)$,

$$\begin{aligned} |a(u, v)| &\leqslant |(u|v)_W| + k\left|\int u(x)\overline{v(x)}\, dx\right| + \left|\int V_2(x)u(x)\overline{v(x)}\, dx\right| \\ &\leqslant \|u\|_W \cdot \|v\|_W + k\|u\|_{L^2(R^n)} \cdot \|v\|_{L^2(R^n)} \\ &\quad + \|V_2\|_{L^{n/2}(R^n)} \cdot \|u\|_{L^{2*}(R^n)} \cdot \|v\|_{L^{2*}(R^n)} \\ &\leqslant c_1\|u\|_W \cdot \|v\|_W + c_2\|V_2\|_{L^{n/2}(R^n)}\|u\|_{H^1(R^n)} \cdot \|v\|_{H^1(R^n)} \\ &\leqslant c_3\|u\|_W \cdot \|v\|_W, \end{aligned}$$

where $2^* = 2n/(n-2)$ and c_1, c_2, c_3 are positive constants. Q.E.D.

THEOREM 2.2. *Let* (i), (ii), (iii) *be satisfied. Then there exists a unique selfadjoint operator* $h\colon D(h) \to L^2(R^n)$ *so that* $D(h) \subset W$ *and* $(hu|v)_{L^2(R^n)} = a(u, v)$ *for* $u \in D(h)$, $v \in W$. *Moreover the spectrum* $\sigma(h)$ *of* h *is formed by a sequence* $\{\lambda_k\}$ *bounded from below of isolated eigenvalues of finite multiplicity and*

$$L^2(R^n) = \sum_k M_k, \qquad M_k \perp M_{k'} \text{ for } k \neq k',$$

where M_k is the eigenmanifold corresponding to λ_k.

Let us initially prove the following

LEMMA 2.3. *There exists $\lambda_0 > 0$ such that for each $\phi \in C_0^\infty(R^n)$:*

$$\alpha\|\phi\|_W^2 \geqslant \int \left\{ |\text{grad}\, \phi(x)|^2 + V(x)|\phi(x)|^2 + \lambda_0|\phi(x)|^2 \right\} dx$$

$$\geqslant \beta\|\phi\|_W^2$$

where α, β are positive constants.

PROOF. $V_2 \in L^{n/2}(R^n)$, therefore (cf. Lemma 3.1 of [8]) for each $\varepsilon > 0$ there exist two functions θ, η such that $\theta \in L^\infty(R^n)$, $\|\eta\|_{L^{n/2}(R^n)} < \varepsilon$ and $V_2 = \theta + \eta$; therefore we have, $\forall\phi \in C_0^\infty(R^n)$,

$$\int \left\{ |\text{grad}\, \phi(x)|^2 + V(x)|\phi(x)|^2 \right\} dx$$

$$= \|\phi\|_W^2 - k\|\phi\|_{L^2(R^n)}^2 + \int (\eta(x) + \theta(x))|\phi(x)|^2\, dx. \qquad (2.1)$$

On the other hand, by virtue of the Sobolev embedding theorem, we have

$$\left| \int (\eta(x) + \theta(x))|\phi(x)|^2\, dx \right|$$

$$\leqslant \|\theta\|_{L^\infty(R^n)} \cdot \|\phi\|_{L^2(R^n)}^2 + \|\eta\|_{L^{n/2}(R^n)} \cdot \|\phi\|_{L^{2^*}(R^n)}^2$$

$$\leqslant \|\theta\|_{L^\infty(R^n)} \cdot \|\phi\|_{L^2(R^n)}^2 + c_1\varepsilon\|\phi\|_{H^1(R^n)}^2$$

$$\leqslant \|\theta\|_{L^\infty(R^n)} \cdot \|\phi\|_{L^2(R^n)}^2 + c_2\varepsilon\|\phi\|_W^2, \qquad (2.2)$$

where c_1, c_2 are positive constants and $2^* = 2n/(n-2)$. From (2.1) and (2.2) the conclusion easily follows if we choose $\varepsilon \leqslant c_2/2$ and $\lambda_0 \geqslant k + \|\theta\|_{L^\infty(R^n)}$. Q.E.D.

Let us now prove Theorem 2.2. If ${u \atop v} \in C_0^\infty(R^n)$ we set

$$a(u, v) + \lambda_0(u|v)_{L^2(R^n)} = [u, v].$$

By virtue of Lemma 2.3, W is isomorphic to the Hilbert space U completion of $C_0^\infty(R^n)$ with respect to the scalar product $[\cdot, \cdot]$; therefore, by (ii) and Theorem 1.1, the embeddings i: $U \hookrightarrow L^2(R^n)$ and its adjoint i^*: $(L^2(R^n))' \hookrightarrow U'$ are compact. Now the proof of the theorem follows from standard arguments (cf., e.g., the proof of Theorem 4.1 of [1] and Lemma II.6 of [7]). Q.E.D.

REMARK 2.4. We observe that (0.2) may be replaced by some other assumption which assures that the sesquilinear form $a(u, v)$ is bounded from below (cf., e.g., [6] and [7]). In fact, if $a(u, v)$ is bounded from below, Lemma 2.3 and, hence, Theorem 2.2 are still valid.

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