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A SAMELSON PRODUCT AND HOMOTOPY-ASSOCIATIVITY

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ABSTRACT. A certain Samelson (commutator) product is computed in the homotopy groups of a finite H-space. This result is applied to the study of H-maps in π_3 of that space, and to the study of multiplications on Lie groups.

0. Introduction. Let m: $S^3 \times S^3 \to S^3$ be a multiplication on the 3-sphere S^3 and let $\alpha \in \pi_3(S^3) \cong Z$ be a generator. Recall that Arkowitz and Curjel [2] have shown that the Samelson (commutator) product $\langle\alpha, \alpha\rangle_m \in \pi_6(S^3) \cong Z/12$ is a generator precisely when m is homotopy-associative. In this paper we shall consider a generalization (Theorem 0.1) of that computation and also present some applications.

We now give specific results. Unless otherwise stated X will denote a finite CW H-space with $\pi_3(X) \cong Z$. Further X must possess at least one homotopy-associative multiplication and satisfy a technical condition 0.4 (which holds for all known finite H-spaces). Included in our study are all compact, connected, simple, nonabelian Lie groups. The following is proved in §1.

0.1 THEOREM. *Let $\alpha \in \pi_3(X)$ be a generator. If μ is any homotopy-associative multiplication on X then $\langle\alpha, \alpha\rangle_\mu$ generates $\pi_6(X)$.*

As an application we consider the question raised at Neuchâtel [9, Problem 24, p. 127] as to whether or not α is an H-map, and prove in §2:

0.2 THEOREM. (a) *Given any multiplication m on S^3 there exists a multiplication μ on X such that α: $(S^3, m) \to (X, \mu)$ is an H-map.*

(b) *Given any multiplication μ on X there exists a multiplication m on S^3 such that α: $(S^3, m) \to (X, \mu)$ is an H-map.*

In a different direction Theorem 0.1 has an application to the study of properties of multiplications on Lie groups. Recall that Mimura [6] has shown that there are $12^{15} \cdot 3^9 \cdot 5 \cdot 7$ distinct multiplications on SU(3). We show in §3

0.3 THEOREM. *At least one-third of the multiplications on* SU(3) *are not homotopy-associative.*

Let us now state the condition we assume (except in §3) all finite H-spaces satisfy in this paper. Let $\tilde{X}$ denote the universal cover of X.

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0.4 CONDITION. The 5-skeleton of $\tilde{X}$ is a bouquet of spaces S^3 and $S^3 \cup_\eta e^5$.

This condition is conjectured true in general by Harper (cf. [4, Proposition 3.4]) and we assume it to be able to make use in §1 of a result from [4].

For the remainder of the introduction we give notation. We work in the homotopy category of path-connected based spaces of the homotopy type of CW-complexes and do not distinguish between maps and homotopy classes of maps. The notation Z/n is used for the integers mod n and G_p and Y_p for the localizations at a prime p of, respectively, an abelian group G and a 1-connected space Y. For basic facts about localizations see [11], and for basic H-space definitions see [8].

1. Proof of Theorem 0.1.

1.1 *Outline of proof.* Let X denote a finite H-space satisfying Condition 0.4 with $\pi_3(X) \cong Z$ generated by α, and let μ be any homotopy-associative multiplication on X. For the purpose of computing $\langle\alpha, \alpha\rangle_\mu \in \pi_6(X)$ we may also assume that X is 1-connected. For if not, the universal cover $\tilde{X}$ of X is again a finite H-space $(\tilde{X}, \tilde{\mu})$ with $\tilde{\mu}$ again homotopy-associative.

Now let P denote Stasheff's projective plane [10] of (X, μ) and let, ambiguously, $\alpha \in \pi_4(P)$ correspond to $\alpha \in \pi_3(X)$. (Of course $\pi_{i+1}(P) \cong \pi_i(X)$ for $i < 10$.) Then since μ is homotopy-associative it is well known that the Samelson product $\langle\alpha, \alpha\rangle_\mu$ corresponds to the Whitehead product $[\alpha, \alpha] \in \pi_7(P)$. We compute this Whitehead product using a technique of Arkowitz [1].

In this direction first note that, if $Y^{[n]}$ denotes the n-Postnikov section of a space Y, the 6-section map $P \to P^{[6]}$ is an isomorphism on homotopy groups in dimensions less than seven. Since $\pi_2(X) = 0$ by [3], the space $P^{[6]}$ has nonvanishing homotopy groups only in (at most) dimensions four, five and six. For these dimensional reasons, then, $P^{[6]}$ is a loop-space. Thus the method of Arkowitz can be applied to the pair $(P^{[6]}, P)$ to give that $[\alpha, \alpha] = \theta(\beta * \beta)$ where β is a generator of $H_4(P^{[6]}) \cong Z$, "$*$" denotes Pontrjagin multiplication in $H_*(P^{[6]})$, and the homomorphism θ is given as $l_*\colon H_8(P^{[6]}) \to H_8(\pi_7(P), 8)$ where $l \in H^8(P^{[6]}; \pi_7(P))$ is the appropriate k-invariant of P.

We now begin detailed calculation of $l_*(\beta * \beta)$ by determining some low dimensional homotopy and homology groups of a 1-connected finite H-space X. By way of notation let $X^{(n)}$ denote the n-skeleton of X.

1.2 *Some homotopy and homology of X.* Set $K = S^3 \cup_\eta e^5$ where η is the essential map $\eta\colon S^4 \to S^3$. Choose a generator $\gamma\colon S^5 \to K$ of the infinite cyclic group$\pi_5(K)$ and set $L = K \cup_\gamma e^6$. By Condition 0.4 and since $\pi_3(X) \cong Z$, the 5-skeleton $X^{(5)}$ of X is either S^3 or K depending on whether $\pi_4(X)$ is $Z/2$ or 0. We next consider the possibilities for $X^{(6)}$. Since $H_3(X)$ is free of rank one the group $H_6(X)$ must be a torsion group, but by [4, Lemma 2.4] the group $H_6(X)$ is always torsion free, hence $H_6(X) = 0$. Thus if $H_5(X)$ has no torsion then $X^{(6)} = X^{(5)}$. In case of torsion then $H_5(X) \cong Z/2$ and $\pi_5(X) =$

0 again by [4, Theorem 1.2]. Therefore $X^{(5)} = K$ and $X^{(6)} = L$. We complete this analysis by noting that the inclusion $X^{(6)} \subset X$ induces an isomorphism on homotopy groups in dimensions less than 6 and an epimorphism in dimension 6. Thus if $X^{(6)} = S^3$ then $\pi_6(X)$ is a quotient of $Z/12$, if $X^{(6)} = K$ it is a quotient of $Z/6$, and if $X^{(6)} = L$ it is a quotient of $Z/3$.

Noticing that at any rate $\pi_6(X)$ is a quotient of $Z/12$ we now divide our calculation of $[\alpha, \alpha]$ into two parts–in 1.3 we show $[\alpha, \alpha]$ generates the 3-torsion, and in 1.4 the 2-torsion, of $\pi_7(P)$. To this end we use the technique of localization. Specifically, equating $\alpha \in \pi_4(P)$ with its localized image in $\pi_4(P_p)$ we have that $[\alpha, \alpha]$ generates the p-torsion in $\pi_7(P)$ if and only if $[\alpha, \alpha]$ generates $\pi_7(P_p)$.

1.3 Lemma. *$[\alpha, \alpha]$ generates the 3-torsion of $\pi_7(P)$.*

Proof. In this proof all spaces have been localized at the prime 3 but if Y denotes any space we continue to write Y for Y_3. Assume, to avoid trivialities, that $Z/3 \subset \pi_7(P)$. Then by 1.2 we have that if $X^{(6)}$ is S^3 or L then $\pi_4(P) \cong Z_3$, $\pi_7(P) \cong Z/3$, and $\pi_i(P) = 0$ for all other $i < 7$. Or if $X^{(6)}$ is K then $\pi_4(P) \cong \pi_6(P) \cong Z_3$, $\pi_7(P) \cong Z/3$, and $\pi_i(P) = 0$ for all other $i < 7$. Thus in the former case $P^{[6]} = K(Z_3, 4)$ and in the latter case, since the only possible k-invariant lies in a zero group, $P^{[6]} = K(Z_3, 4) \times K(Z_3, 6)$. In either case, if β is a generator of $H_4(P^{[6]}) \cong Z_3$–that is, $3 \nmid \beta$–then $\beta * \beta = 2\gamma_2(\beta)$ where γ_2: $H_4(P^{[6]}) \cong H_4(Z_3, 4) \to H_8(Z_3, 4) \cong H_8(P^{[6]})$ is the second divided power in the ring $H_*(Z_3, 4)$.

Now let $y \in H^4(P^{[6]}; Z/3) \cong Z/3$ be the basic class. We show in §4 that the k-invariant $l \in H^8(P^{[6]}; Z/3)$ is always given by $l = a\mathcal{P}^1 y + by^2$ where $3 \nmid a$ and $3 \nmid b$. Then by 1.1

$$\begin{aligned}[\alpha, \alpha] &= l_*(\beta * \beta) = (a\mathcal{P}^1 y + by^2)_*(\beta * \beta) \\ &= (a\mathcal{P}^1 y)_*(\beta * \beta) + (by^2)_*(\beta * \beta)\end{aligned}$$

(this since if f and g are maps $A \to B$ where B is an n-connected H-space, then $(f + g)_* = f_* + g_*$ on homology in dimensions less than $2n + 2$)

$$= (a\mathcal{P}^1 y)_*(\beta) * (a\mathcal{P}^1 y)_*(\beta) + (by^2)_*(\beta * \beta)$$

(this since $\mathcal{P}^1 y$ is primitive)

$$= (by^2)_*(\beta * \beta)$$

(this since $(\mathcal{P}^1 y)_*(\beta) \in H_4(Z/3, 8) = 0$)

$$= 2by_*^2(\gamma_2(\beta)) \text{ which is a generator of } \pi_7(P)$$

since $y_*^2(\gamma_2(\beta))$ is a generator of $H_8(Z/3, 8) \cong \pi_7(P)$. Q.E.D.

1.4 Lemma. *$[\alpha, \alpha]$ generates the 2-torsion of $\pi_7(P)$.*

Proof. In this proof all spaces have been localized at the prime 2 but if Y denotes any space we continue to write Y for Y_2. Again to avoid trivialities assume $Z/2 \subset \pi_7(P)$. By 1.2 we need only consider the two cases $X^{(6)} = S^3$ or $X^{(6)} = K$.

It may be determined in the first case that $H_8(P^{[6]}) \cong Z_2 \oplus Z/4$ and in the second case that $H_8(P^{[6]}) \cong Z_2 \oplus Z/2$. Furthermore in both cases if β is a generator of $H_4(P^{[6]}) \cong Z_2$–that is, $2 \nmid \beta$–then $\beta * \beta = 2\gamma_2(\beta) + u \in H_8(P^{[6]})$ where γ_2: $H_4(P^{[6]}) \cong H_4(Z_2, 4) \to H_8(Z_2, 4) \cong Z_2 \subset H_8(P^{[6]})$ is the second divided power in the ring $H_*(Z_2, 4)$ and where u is a generator of the appropriate $Z/4$ or $Z/2$ summand in $H_8(P^{[6]})$. We omit calculation of these facts.

We have left to determine the k-invariant $l \in H^8(P^{[6]}; \pi_7(P))$. Suppose $\pi_7(P) \cong Z/2$; we consider the remaining case $\pi_7(P) \cong Z/4$ later. Then $H^8(P^{[6]}; Z/2)$ is isomorphic to $\operatorname{Hom}(H_8(P^{[6]}), Z/2) \cong Z/2 \oplus Z/2$ with generators z and x such that $z_*(\gamma_2(\beta)) = w$, $z_*(u) = 0$ and $x_*(\gamma_2(\beta)) = 0$, $x_*(u) = w$ where w generates $Z/2 \cong H_8(Z/2, 8)$. It follows that z is the element y^2 where $y \in H^4(P^{[6]}; Z/2) \cong Z/2$ is the basic class.

Thus the k-invariant l is either $0, y^2, y^2 + x$, or x. An easy calculation rules out 0 and y^2 as choices for l and in each of the two remaining cases we may now compute that $[\alpha, \alpha] = l_*(\beta * \beta) = x_*(2\gamma_2(\beta) + u) = w$ which generates $H_8(Z/2, 8) \cong \pi_7(P)$.

Finally we have left to consider the case $\pi_7(P) \cong Z/4$. We can show that $\langle \alpha, \alpha \rangle_\mu$ generates the 2-torsion of $\pi_6(X)$ directly in this case by noting that here the 6-Postnikov section $X^{[6]} = (S^3)^{[6]}$ (recall all spaces are 2-local). For then it is known [**2**, Lemma 4(c)] that, with respect to any multiplication m on S^3, $\langle \gamma, \gamma \rangle_m$ generates the 2-torsion of $\pi_6(S^3)$ for any generator $\gamma \in \pi_3(S^3)$. This concludes the proof of 1.4 and with it the proof of Theorem 0.1.

2. Proof of Theorem 0.2. Recall our Condition 0.4 on X, and that X possesses at least one homotopy-associative multiplication, and that $\pi_3(X) \cong Z$ with generator α. We shall need the following corollary of Theorem 0.1: Let $S = S^3$ throughout this section.

2.1 Lemma. *The homomorphism* $\alpha \wedge \alpha^*$: $[X \wedge X, X] \to [S \wedge S, X]$ *is an epimorphism.*

Proof. If 1: $X \to X$ is the identity map then $\alpha \wedge \alpha^* \langle 1, 1 \rangle_\mu = \langle \alpha, \alpha \rangle_\mu$, which, by 0.2, generates $[S \wedge S, X] \cong \pi_6(X)$ for any homotopy-associative multiplication μ. Q.E.D.

2.2 Proof of Theorem 0.2 (a). Fix any homotopy-associative multiplication μ_0 on X. Then any multiplication μ on X is given by $\mu = \mu_0 \cdot q^*(\eta)$ for some map η: $X \wedge X \to X$ and where q: $X \times X \to X \wedge X$ denotes projection and where the multiplication "$\cdot$" is taken with respect to μ_0.

We must find an η such that

$$\alpha \circ m = \mu_0 \cdot q^*(\eta) \circ \alpha \times \alpha \Leftrightarrow$$
$$\alpha \circ m = (\mu_0 \circ \alpha \times \alpha) \cdot (q^*(\eta) \circ \alpha \times \alpha). \tag{2.3}$$

But the map $(\mu_0 \circ (\alpha \times \alpha))^{-1} \cdot (\alpha \circ m)$: $S \times S \to X$ is null-homotopic when restricted to $S \vee S$. Hence there is a map θ: $S \wedge S \to X$ such that $p^*(\theta) = (\mu_0 \circ \alpha \times \alpha)^{-1} \cdot (\alpha \circ m)$ where p: $S \times S \to S \wedge S$ denotes projection.

Thus it suffices to find η such that $p^*(\theta) = q^*(\eta) \circ \alpha \times \alpha$, which is equivalent to requiring $p^*(\theta) = p^*(\alpha \wedge \alpha^*(\eta))$. Finally since $\alpha \wedge \alpha^*$ is epimorphic by Lemma 2.1, there exists an $\eta \in [X \wedge X, X]$ so that $\theta = \alpha \wedge \alpha^*(\eta)$. Q.E.D.

The next lemma follows easily from the discussion in 1.2 and is given without proof.

2.4 Lemma. *The homomorphism* α_*: $\pi_6(S) \to \pi_6(X)$ *is an epimorphism.*

2.5 Proof of Theorem 0.2 (b). Fix any multiplication m_0 on S. Then by Theorem 0.2 (a) there exists a multiplication μ_0 on X such that α: $(S, m_0) \to (X, \mu_0)$ is an H-map. Now write the given multiplication μ as $\mu = \mu_0 \cdot q^*(\eta)$ for some choice of η: $X \wedge X \to X$.

Our task is to find a map σ: $S \wedge S \to S$ and thus a multiplication $m = m_0 \cdot p^*(\sigma)$ on S so that

$$\alpha \circ m_0 \cdot p^*(\sigma) = \mu_0 \cdot q^*(\eta) \circ \alpha \times \alpha \tag{2.6}$$

(the multiplication "$\cdot$" in $m_0 \cdot p^*(\sigma)$ is taken with respect to m_0).

Because we chose μ_0 to make α: $(S, m_0) \to (X, \mu_0)$ an H-map, the equation (2.6) is equivalent to

$$(\alpha \circ m_0) \cdot (\alpha \circ p^*(\sigma)) = (\mu_0 \circ \alpha \times \alpha) \cdot (q^*(\eta) \circ \alpha \times \alpha). \tag{2.7}$$

Now by **[5]** for any H-space (X, μ_0) the loop $[S \times S, X]$ is in fact a group. Therefore, again since α is an H-map, the equation (2.7) is equivalent to $\alpha \circ p^*(\sigma) = q^*(\eta) \circ \alpha \times \alpha$ which is equivalent to $p^*(\alpha_*(\sigma)) = p^*(\alpha \wedge \alpha^*(\eta))$. Finally since α_*: $\pi_6(S) \to \pi_6(X)$ is epimorphic by Lemma 2.4, there exists a $\sigma \in \pi_6(S)$ so that $\alpha_*(\sigma) = \alpha \wedge \alpha^*(\eta)$. Q.E.D.

3. Homotopy-associativity in SU(3). In this section we shall prove Theorem 0.3. We begin with a simple lemma on Samelson products whose proof we omit. Consider an arbitrary homotopy-abelian H-space (Y, m_0). For the purpose of §3 only we do not place any restrictions on the space Y; in particular Y is not required to be a finite complex. For any $\eta \in [Y \wedge Y, Y]$ denote the multiplication m_η on Y by $m_\eta = q^*(\eta) + m_0$ where "+" is multiplication in $[Y \times Y, Y]$ with respect to m_0 written additively, and q: $Y \times Y \to Y \wedge Y$ denotes projection.

3.1 Lemma. *Let* α *be an arbitrary element in* $\pi_3(Y)$. *Then the Samelson product* $\langle \alpha, \alpha \rangle_{m_\eta} = 0$ *if and only if* $2(\alpha \wedge \alpha)^*(\eta) = 0$ *in* $\pi_6(Y)$.

3.2 Proof of Theorem 0.3. Let Y denote the 3-localization of the Lie group SU(3); Y is easily seen to possess a homotopy-abelian multiplication m_0. Let $\alpha \in \pi_3(Y) \cong Z_3$ be such that $3 \nmid \alpha$ and consider the function $(\alpha \wedge \alpha)^*$: $[Y \wedge Y, Y] \to \pi_6(Y) \cong Z/3$. We consider $(\alpha \wedge \alpha)^*$ a homomorphism of groups with respect to the 3-localization of the standard Lie group multiplication on SU(3).

Now suppose $\eta \in \ker(\alpha \wedge \alpha)^*$. Then on one hand, by Lemma 3.1, $\langle \alpha, \alpha \rangle_{m_\eta} = 0$. But on the other hand it follows from Theorem 0.1 that if $m_\eta = q^*(\eta) + m_0$ is homotopy-associative then $\langle \alpha, \alpha \rangle_{m_\eta} \neq 0$. Therefore if $\eta \in \ker(\alpha \wedge \alpha)^*$ then m_η is not homotopy-associative. By Mimura [6] it follows that $|[Y \wedge Y, Y]| = 3^9$ so that $|\ker(\alpha \wedge \alpha)^*| = 3^8$ by Lemma 2.1. Theorem 0.3 is now a consequence of the fact that a multiplication μ on a space X is homotopy-associative only if it is again homotopy-associative when localized at any prime p.

4. The k-invariant l. We employ the notation and conventions of §1.3 (in particular all spaces are 3-local). In this section we compute the k-invariant $l \in H^8(P^{[6]}; \pi_7(P))$ to be $l = a\mathcal{P}^1 y + by^2$ where y is a generator of $H^4(P^{[6]}; \pi_7(P))$ and $3 \nmid a$ and $3 \nmid b$.

The k-invariant is obtained (see for example [7, Chapter 13]) by considering the 6-Postnikov section map $g\colon P \to P^{[6]}$ and letting F denote the fibre of g to get the fibration $F \to P \to P^6$. Then l is given from the Serre exact sequence of this fibration

$$H^7(F; \pi_7(P)) \xrightarrow{\tau} H^8\big(P^{[6]}; \pi_7(P)\big) \xrightarrow{g^*} H^8(P; \pi_7(P)) \tag{4.1}$$

by $l = \tau(i)$ where τ is transgression and i is the basic class in $H^7(F; \pi_7(P))$.

From the possibilities for P and $P^{[6]}$ determined in §1 we have that $\pi_7(P) \cong Z/3$ and $H^8(P^{[6]}; Z/3) \cong Z/3 \oplus Z/3$ with generators $\mathcal{P}^1 y$ and y^2 where y is a generator of $H^4(P^{[6]}; Z/3) \cong Z/3$. Then $H^7(F; Z/3) \cong Z/3$ and the computation of l will follow easily if we can show that $\mathcal{P}^1 g^*(y)$ and $g^*(y)^2$ are linearly dependent in $H^8(P; Z/3)$.

To this end first note that since μ is homotopy-associative the third projective space $XP(3)$ of (X, μ) exists. From now on write $H^*(\ )$ for $H^*(\ ; Z/3)$ and write $x = g^*(y) \in H^4(P) \cong H^4(XP(3))$. It is known that $0 \neq x^3 \in H^{12}(XP(3))$. Then since $\mathcal{P}^2 x = x^3 \neq 0$ and since $\mathcal{P}^1\mathcal{P}^1 = -\mathcal{P}^2$ we have that $0 \neq \mathcal{P}^1 x \in H^8(P)$.

Now consider the $H^*(\ )$ spectral sequence of the fibration $X \to Q \to P$ where Q denotes the 3-fold join of X. Thereby obtain the exact sequence $E_4^{4,3} \to E_4^{8,0} \to E_5^{8,0} \to 0$ which can be identified with the exact sequence

$$H^4(P) \otimes H^3(X) \xrightarrow{\lambda} H^8(P) \xrightarrow{\rho} E_\infty^{8,0} \to 0. \tag{4.2}$$

Here $\lambda(d \otimes e) = d \cup \tau(e)$ for any $d \in H^4(P)$ and $e \in H^3(X)$, and $\rho(w)$ is defined to be the class of $w \in H^8(P) \cong E_2^{8,0}$ in the quotient group $E_\infty^{8,0}$.

Let z be a generator of $H^3(X)$ such that $\tau(z) = x = g^*(y)$. Then by naturality $\rho(\mathcal{P}^1 x) = \rho(\mathcal{P}^1\tau(z)) = \rho(\tau(\mathcal{P}^1 z))$. But by [4, Theorem 1.4], the operation $\mathcal{P}^1$ is zero on $H^3(X)$ since $\pi_6(X) \cong Z/3$. So $\rho(\mathcal{P}^1 x) = \rho(\tau(0)) = 0$. Therefore by (4.2), $0 \neq \mathcal{P}^1 x = \lambda(v)$ for some $v \in H^4(P) \otimes H^3(X) \cong Z/3$. Finally by writing v as $c(x \otimes z)$ for some c where $3 \nmid c$ we conclude that $\mathcal{P}^1 x = cx^2$.

We finish the computation of l. Choose c as above and obtain $g^*(\mathcal{P}^1 y - cy^2) = \mathcal{P}^1 x - cx^2 = 0$. By the exactness of (4.1) there is a $u \in H^7(F; Z/3) \cong$

$Z/3$ so that $\tau(u) = \mathcal{P}^1 y - cy^2$. But the basic class $i = ku$ for some k where $3 \nmid k$. Therefore $l = \tau(i) = k\mathcal{P}^1 y - kcy^2$. We are done by setting $a = k$ and $b = -kc$. Q.E.D.

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