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INNER AMENABILITY AND CONJUGATION OPERATORS

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ABSTRACT. It is shown that an infinite discrete group G is inner amenable if and only if the C^*-algebra generated by the unitaries on $l^2(G)$ corresponding to conjugation by s ($s \in G$) does not contain the projection on the point-mass at the identity.

Let G be an infinite discrete group with identity e, and let $s \to L_s$ (resp. R_s) denote the left (resp. right) regular representation of G on $H = l^2(G)$. For $s \in G$, let $U_s = L_s R_s$, so $U_s\xi(t) = \xi(s^{-1}ts)$ for $\xi \in H$. Write $C^*(L_G, R_G)$ for the C^*-algebra generated by the unitaries L_s, R_s ($s \in G$), and $C^*(U_G)$ for the C^*-subalgebra of $C^*(L_G, R_G)$ generated by the unitaries U_s. Let δ denote the characteristic function of $\{e\}$, and P_δ the projection on the one-dimensional subspace of H spanned by δ. In [2], using computations from [1], C. A. Akemann and P. A. Ostrand proved that $C^*(L_G, R_G)$ contains the compact operators when G is the free group on two generators by showing that in this case one has $P_\delta \in C^*(U_G)$. Our theorem below provides an easier proof (and a generalization) of this result.

Following E. G. Effros [3], we say that G is *inner amenable* if there is a state m on the C^*-algebra $l^\infty(G)$ such that $m(\delta) = 0$ and m is invariant under the automorphisms T_s ($s \in G$) of $l^\infty(G)$ defined by $(T_sf)(t) = f(s^{-1}ts)$. Such an m is called a nontrivial *inner mean* on G. Inner amenability is a considerably weaker condition on G than amenability in the usual sense. The free group on two generators is an easily accessible example of a group which is not inner amenable (see [3]). Inner amenability and the behavior of $C^*(U_G)$ are related by the following theorem.

THEOREM. *The group G is inner amenable if and only if $P_\delta \notin C^*(U_G)$.*

PROOF. First suppose that $P_\delta \notin C^*(U_G)$. Since $U_s\delta = \delta$ for each $s \in G$, it follows that $P_\delta X = XP_\delta = (X\delta, \delta)P_\delta$ for every $X \in C^*(U_G)$, so $C^*(U_G) + \mathbf{C}P_\delta$ is a *-algebra. We claim that it is norm-closed, and hence a C^*-algebra. [Suppose $X_n + z_nP_\delta \to Y$ in norm, with $X_n \in C^*(U_G)$ and $z_n \in \mathbf{C}$. The sequence $\{z_n\}$ must be bounded (for otherwise we could pass to a subsequence and assume $|z_n| \to \infty$, forcing $z_n^{-1}X_n + P_\delta \to 0$ and thereby contradicting $P_\delta \notin C^*(U_G)$), so we may assume that $z_n \to z \in \mathbf{C}$ and hence

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$X_n \to Y - zP_\delta$. We conclude that $Y \in C^*(U_G) + \mathbf{C}P_\delta$.] Our hypothesis permits us to define a linear functional ϕ on $C^*(U_G) + \mathbf{C}P_\delta$ by $\phi(X + zP_\delta) = (X\delta, \delta)$. Notice that

$$\phi((X + zP_\delta)^*(X + zP_\delta)) = (X^*X\delta, \delta) \geqslant 0,$$

so ϕ is a state. Extend ϕ to a state ψ on the algebra $B(H)$ of bounded operators on H. Since $\psi(U_s) = 1$, we see that $I - U_s$ belongs to the left and right kernels of ψ for every $s \in G$ and hence $\psi(U_s X U_s^*) = \psi(X)$ for every $X \in B(H)$. For $f \in l^\infty(G)$, let $\pi(f)$ be the corresponding multiplication operator on H. Since $U_s \pi(f) U_s^* = \pi(T_s f)$ and $\pi(\delta) = P_\delta$, the state $\psi \circ \pi$ on $l^\infty(G)$ is a nontrivial inner mean for G, as required.

For the converse direction, suppose that G is inner amenable. The "convergence to invariance" argument in [3] yields a net $\{\xi_\alpha\}$ of unit vectors in H with $\xi_\alpha(e) = 0$ for every α and

$$\lim_\alpha \|U_s\xi_\alpha - \xi_\alpha\| = 0$$

for every $s \in G$. Let ψ be any w^*-limit state of the net of vector states on $B(H)$ corresponding to the net $\{\xi_\alpha\}$. We have $\psi(U_s) = 1$ $(s \in G)$ and $\psi(P_\delta) = 0$. By applying ψ and the vector state corresponding to δ, one sees that the norm of the difference of P_δ and any finite linear combination of the U_s's must be at least $\frac{1}{2}$, so $P_\delta \notin C^*(U_G)$.

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