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BOUNDARY REPRESENTATIONS AND TENSOR PRODUCTS OF C^*-ALGEBRAS[1]

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ABSTRACT. Let A and B be unital, generating linear subspaces of C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. If either $\mathcal{A}$ or $\mathcal{B}$ is a GCR algebra, then the set of boundary representations for $A \otimes B$ can be identified with the Cartesian product of the boundary representations for A with the boundary representations for B.

Let $\mathcal{A}$ be a unital C^*-algebra and let A be a linear subspace of $\mathcal{A}$ which contains the unit, 1, and which generates $\mathcal{A}$ as a C^*-algebra. An irreducible representation π of $\mathcal{A}$ is said to be a *boundary representation* for A if π is the only completely positive extension to $\mathcal{A}$ of the restriction $\pi|A$. (A linear mapping ϕ on $\mathcal{A}$ is *completely positive* if $\phi \otimes 1_n$ is positive for all $n = 1, 2, \ldots,$ where 1_n is the identity mapping on the algebra of $n \times n$ complex matrices.) Boundary representations were introduced by Arveson in **[1]**, where he demonstrated the rôle played by boundary representations in determining the extent to which the order and norm structure on a unital linear space, A, of operators determines $C^*(A)$, the C^*-algebra generated by A.

If $\mathcal{A}$ is a commutative C^*-algebra, a boundary representation is essentially just a point in the Choquet boundary of A. If $\mathcal{A}$ is commutative, then $\mathcal{A} = C(X)$ for some compact Hausdorff space X and each irreducible representation π of $\mathcal{A}$ is just point evaluation at some point $x \in X$. Every unital positive linear mapping on $\mathcal{A}$ is completely positive and is also just integration with respect to some probability measure μ on X. The assertion that π is the only completely positive extension to $\mathcal{A}$ of $\pi|A$ becomes the assertion that point mass at x is the only probability measure μ on X for which

$$\int f(y)d\mu(y) = f(x),$$

for all $f \in A$. But this just says that x lies in the Choquet boundary for A. See **[4]** for more details.

If A is a unital, generating subspace of $\mathcal{A}$, let bd(A) denote the set of boundary representations for A. Suppose that B is a unital, generating subspace of a second C^*-algebra $\mathcal{B}$. Let $\mathcal{A} \otimes \mathcal{B}$ denote the algebraic tensor

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product of $\mathcal{A}$ and $\mathcal{B}$ and let $\mathcal{A} \otimes_\gamma \mathcal{B}$ denote the closure of $\mathcal{A} \otimes \mathcal{B}$ when the latter is provided with the C^*-cross norm γ. The algebraic tensor product $A \otimes B$ of A and B is a unital, generating subspace of $\mathcal{A} \otimes_\gamma \mathcal{B}$ and we may ask about the relation between $\mathrm{bd}(A \otimes B)$ and the two sets $\mathrm{bd}(A)$ and $\mathrm{bd}(B)$.

The question is easy to answer in the commutative case. If $\mathcal{A} = C(X)$ and $\mathcal{B} = C(Y)$ then $\mathcal{A} \otimes_\gamma \mathcal{B} = C(X \times Y)$ and

$$\mathrm{bd}(A \otimes B) = \mathrm{bd}(A) \times \mathrm{bd}(B).$$

In this paper we shall show that the same result holds anytime one of the factors is a GCR algebra. In the case in which $\mathcal{B}$ is the algebra, M_n, of $n \times n$ matrices, this result is essentially contained in **[3]**.

If $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, if γ is any C^*-cross norm on $\mathcal{A} \otimes \mathcal{B}$, and if π_1 (resp. π_2) is an irreducible representation of $\mathcal{A}$ (resp. $\mathcal{B}$), then $\pi_1 \otimes_\gamma \pi_2$ is an irreducible representation of $\mathcal{A} \otimes_\gamma \mathcal{B}$. We first show that if $\pi_1 \otimes_\gamma \pi_2$ is a boundary representation, then so are π_1 and π_2.

LEMMA 1. *Suppose that A is a unital, generating subspace for $\mathcal{A}$, that B is a unital, generating subspace for $\mathcal{B}$ and that $\pi_1 \otimes_\gamma \pi_2$ is a boundary representation for $A \otimes B$. Then $\pi_1 \in \mathrm{bd}(A)$ and $\pi_2 \in \mathrm{bd}(B)$.*

PROOF. Suppose the contrary. Then one of the factors, say π_1, is not a boundary representation. Hence there exists a completely positive linear map $\phi_1: \mathcal{A} \to \mathcal{L}(\mathcal{H}_1)$ such that $\phi_1 \neq \pi_1$ but $\phi_1|A = \pi_1|A$. (Here, $\mathcal{H}_1$ is the Hilbert space on which π_1 acts.) By means of the Stinespring representation for completely positive maps **[6]**, one can show that the tensor product of two completely positive maps is again completely positive. Therefore $\phi_1 \otimes_\gamma \pi_2$ is a completely positive extension of $\pi_1 \otimes_\gamma \pi_2|A \otimes B$ which is unequal to $\pi_1 \otimes_\gamma \pi_2$. This contradicts the assumption that $\pi_1 \otimes_\gamma \pi_2 \in \mathrm{bd}(A \otimes B)$.

In general, not every irreducible representation π on $\mathcal{A} \otimes_\gamma \mathcal{B}$ factors as a product $\pi_1 \otimes_\gamma \pi_2$ of irreducible representations; if, however, we assume that one of the C^*-algebras is a GCR algebra, then every irreducible representation does factor **[2]**. Note, also, that since GCR algebras are nuclear, there is a unique C^*-crossnorm on $\mathcal{A} \otimes \mathcal{B}$, which we denote by $\mathcal{A} \otimes_m \mathcal{B}$. Thus, when one of the factors is GCR, Lemma 1 asserts $\mathrm{bd}(A \otimes B) \subseteq \mathrm{bd}(A) \times \mathrm{bd}(B)$, provided that we identify the pair (π_1, π_2) with the product $\pi_1 \otimes_m \pi_2$.

The following lemma is probably known, but no reference for it could be found.

LEMMA 2. *Let $\mathcal{S}$ be a unital C^*-algebra contained in another C^*-algebra $\mathcal{T}$ (with the same unit). Let ϕ be a completely positive map on $\mathcal{T}$, and let π be a representation of $\mathcal{S}$ such that $\phi|_{\mathcal{S}} = \pi$. Then $\phi(ts) = \phi(t)\pi(s)$ and $\phi(st) = \pi(s)\phi(t)$, for all $s \in \mathcal{S}$ and $t \in \mathcal{T}$.*

PROOF. Let $\phi = V^*\sigma V$ be the Stinespring representation for ϕ (see **[6]**), and let $P = VV^*$ be the range projection of V. Observe that $P\sigma P$ restricted to $\mathcal{S}$ is unitarily equivalent to $\phi|_{\mathcal{S}} = \pi$. In particular, $P\sigma P$ is multiplicative on $\mathcal{S}$, so that P is semi-invariant for $\sigma(\mathbb{S})$, i.e. P is the difference between a pair of nested invariant projections for $\sigma(\mathbb{S})$. (See [5].) Since $\sigma(\mathbb{S})$ is a C^*-algebra, P is in fact reducing for $\sigma(\mathbb{S})$; thus P commutes with $\sigma(\mathbb{S})$. For any $s \in \mathbb{S}$ and $t \in \mathfrak{T}$ we then have

$$\begin{aligned}\phi(st) &= V^*\sigma(st)V = V^*\sigma(s)\sigma(t)V\\ &= V^*P\sigma(s)\sigma(t)V = V^*\sigma(s)P\sigma(t)V\\ &= V^*\sigma(s)VV^*\sigma(t)V = \phi(s)\phi(t) = \pi(s)\phi(t).\end{aligned}$$

A similar equality proves $\phi(ts) = \phi(t)\pi(s)$.

As an immediate consequence of this lemma we have the following:

COROLLARY. *Let π_1 and π_2 be representations of C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let ϕ be a completely positive map defined on $\mathcal{A} \otimes_\gamma \mathcal{B}$ and assume that $\phi(1 \otimes b) = 1 \otimes \pi_2(b)$, for all $b \in \mathcal{B}$. Then*

$$\phi(a \otimes b) = \phi(a \otimes 1)(1 \otimes \pi_2(b)) = (1 \otimes \pi_2(b))\phi(a \otimes 1),$$

for all $a, b \in \mathcal{B}$. If, in addition, we assume that $\phi(a \otimes 1) = \pi_1(a) \otimes 1$, for all $a \in \mathcal{A}$, then $\phi = \pi_1 \otimes_\gamma \pi_2$.

LEMMA 3. *Let A and B be unital, generating subspaces of C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\pi_1 \in \operatorname{bd}(A)$ and $\pi_2 \in \operatorname{bd}(B)$. Then $\pi_1 \otimes_\gamma \pi_2 \in \operatorname{bd}(A \otimes B)$.*

PROOF. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the Hilbert spaces on which π_1 and π_2 act. Then $\pi_1 \otimes_\gamma \pi_2$ acts on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Let ϕ be a completely positive linear mapping of $\mathcal{A} \otimes_\gamma \mathcal{B}$ into $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ such that $\phi(x) = \pi_1 \otimes_\gamma \pi_2(x)$, for all $x \in A \otimes B$. We must prove that $\phi = \pi_1 \otimes_\gamma \pi_2$. By the corollary to Lemma 2, it suffices to prove that ϕ agrees with $\pi_1 \otimes_\gamma \pi_2$ on each of the subalgebras $1 \otimes \mathcal{B}$ and $\mathcal{A} \otimes 1$.

Let E be any rank one projection in $\mathcal{L}(\mathcal{H}_2)$. The mapping

$$a \to (1 \otimes E)\phi(a \otimes 1)(1 \otimes E)$$

is the composition of three completely positive mappings and hence is itself a completely positive mapping defined on $\mathcal{A}$. Let e be a unit vector in the range of E and let $\mathcal{H}$ be the range of $1 \otimes E$. Then the transformation U defined by $Ux = x \otimes e$, $x \in \mathcal{H}_1$, is a unitary mapping of $\mathcal{H}_1$ onto $\mathcal{H}$. Let $\hat{\pi}(a) = U\pi_1(a)U^*$, $a \in \mathcal{A}$. Note that $\hat{\pi}(a)$ is the restriction to $\mathcal{H}$ of

$$\pi_1(a) \otimes E = (1 \otimes E)(\pi_1(a) \otimes 1)(1 \otimes E).$$

Since $\hat{\pi}$ is unitarily equivalent to π_1, $\hat{\pi} \in \operatorname{bd}(A)$. Let $\psi(a)$ be the restriction to $\mathcal{H}$ of $(1 \otimes E)\phi(a \otimes 1)(1 \otimes E)$. Then ψ is completely positive and agrees with $\hat{\pi}$ on A, hence on all of $\mathcal{A}$.

Let $x, y \in \mathcal{H}_1$ and $r \in \mathcal{H}_2$. The paragraph above asserts that, for any $a \in \mathcal{A}$,

$$\langle \phi(a \otimes 1)(x \otimes r), y \otimes r \rangle = \langle (\pi_1(a) \otimes 1)(x \otimes r), y \otimes r \rangle.$$

(Just let E be the rank one projection on the subspace spanned by r.) Let

$D = \phi(a \otimes 1) - \pi_1(a) \otimes 1$. So, we have $\langle D(x \otimes r), y \otimes r\rangle = 0$, for all x, $y \in \mathcal{H}_1$, $r \in \mathcal{H}_2$. The polarization formula

$$\begin{aligned} 4\langle D(x \otimes r), y \otimes s\rangle = & \left\langle D(x \otimes (r+s)), y \otimes (r+s)\right\rangle \\ & - \left\langle D(x \otimes (r-s)), y \otimes (r-s)\right\rangle \\ & + i\left\langle D(x \otimes (r+is)), y \otimes (r+is)\right\rangle \\ & - i\left\langle D(x \otimes (r-is)), y \otimes (r-is)\right\rangle \end{aligned}$$

yields $\langle D(x \otimes r), y \otimes s\rangle = 0$, for all x, $y \in \mathcal{H}_1$ and all r, $s \in \mathcal{H}_2$. Consequently, if $z_1 = \Sigma_1^n x_i \otimes r_i$ and $z_2 = \Sigma_1^m y_i \otimes s_i$, then $\langle Dz_1, z_2\rangle = 0$. Since z_1, z_2 run through a dense subset of $\mathcal{H}_1 \otimes \mathcal{H}_2$ and D is bounded, we obtain $D = 0$. Thus $\phi(a \otimes 1) = \pi_1(a) \otimes 1$, for all $a \in \mathcal{A}$. The equality $\phi(1 \otimes b) = 1 \otimes \pi_2(b)$, for all $b \in \mathcal{B}$ is obtained in the same way. This proves the lemma.

If we keep in mind the fact that if either $\mathcal{A}$ or $\mathcal{B}$ is a GCR algebra then any irreducible representation of $\mathcal{A} \otimes_m \mathcal{B}$ is the tensor product of irreducible representations of $\mathcal{A}$ and $\mathcal{B}$ we obtain the following theorem.

THEOREM. *Let A and B be unital, generating subspaces of C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Assume that either $\mathcal{A}$ or $\mathcal{B}$ is a GCR algebra. Then* $\operatorname{bd}(A \otimes B) = \operatorname{bd}(A) \times \operatorname{bd}(B)$.

BIBLIOGRAPHY

1. W. Arveson, *Subalgebras of C^*-algebras*, Acta Math. **123** (1969), 141–224.

2. A. Guichardet, *Tensor products of C^*-algebras*, Aarhus University Lecture Notes Series, No. 12, Aahrus, 1969.

3. A. Hopenwasser, *Boundary representations on C^*-algebras with matrix units*, Trans. Amer. Math. Soc. **177** (1973), 483–490.

4. R. Phelps, *Lectures on Choquet's theorem*, Van Nostrand, Princeton, N. J., 1966.

5. D. Sarason, *On spectral sets having connected complement*, Acta Sci. Math. (Szeged) **26** (1956), 289–299.

6. W. Stinespring, *Positive functions on C^*-algebras*, Proc. Amer. Math. Soc., **6** (1955), 211–216.

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