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SUBMETRIZABLE SPACES AND ALMOST σ-COMPACT FUNCTION SPACES

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ABSTRACT. It is shown that the space of real-valued continuous functions, with the compact-open topology, defined on a Urysohn k-space X is almost σ-compact if and only if X is submetrizable.

A topological space X has a metrizable compression if its topology contains some metrizable topology on X. Such a space is also called submetrizable. A similar definition can be given for a space having a separable metrizable compression. A space is almost σ-compact if it contains a dense σ-compact subspace. Note that every separable space is almost σ-compact. These concepts play an interesting role in the theory of functions spaces, as the following theorem found in **[4]** illustrates. The notation $C_\pi(X)$ and $C_\kappa(X)$ are used to denote the space of real-valued continuous functions on X with the topology of pointwise convergence and the compact-open topology, respectively.

THEOREM 1. *Let X be a completely regular space.*

(a) *$C_\pi(X)$ is separable if and only if X has a separable metrizable compression.*

(b) *$C_\pi(X)$ has a separable metrizable compression if and only if X is separable.*

(c) *$C_\kappa(X)$ has a metrizable compression if and only if X is almost σ-compact.*

In this paper we prove a theorem which is dual to statement (c) of Theorem 1, just as (a) is dual to (b). This dual to (c) is: $C_\kappa(X)$ is almost σ-compact if and only if X has a metrizable compression. Unfortunately, the "only if" part of this statement requires that X be a k-space. This will be illustrated with an example. We shall use in our proof two well-known theorems. The versions of these theorems which we need are the following, and can be found in **[3]** and **[2]**.

THEOREM 2 (ASCOLI THEOREM). *Let F be a closed subset of $C_\kappa(X)$.*

(a) *If F is pointwise bounded and equicontinuous, then F is compact.*

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(b) *If X is a Hausdorff k-space and F is compact, then F is pointwise bounded and equicontinuous.*

THEOREM 3 (STONE-WEIERSTRASS THEOREM). *Every subalgebra of $C_\kappa(X)$ which contains a nonzero constant function and separates points is dense in $C_\kappa(X)$.*

First we take a submetrizable space and show that the space of real-valued continuous functions on it is almost σ-compact. This can be proved for any topology on the function space which is larger than or equal the topology of pointwise convergence and smaller than or equal the compact-open topology. However, we only consider the compact-open topology here.

THEOREM 4. *If X is submetrizable, then $C_\kappa(X)$ is almost σ-compact.*

PROOF. Let $\varphi: X \to Z$ be a continuous injection into metric space Z. The induced function $\varphi^*: C_\kappa(Z) \to C_\kappa(X)$, defined by $\varphi^*(f) = f \circ \varphi$ for every $f \in C_\kappa(Z)$, is continuous and its image is dense in $C_\kappa(X)$. Since almost σ-compactness is preserved by continuous functions, we need only show that $C_\kappa(Z)$ is almost σ-compact.

For each $n \in \mathbf{N}$ ($\mathbf{N}$ is the set of natural numbers), let $\mathfrak{B}_n$ be a locally finite open refinement of the family of all open $(1/n)$-balls in Z, and let F_n be a partition of unity subordinated to $\mathfrak{B}_n$; we may consider F_n as a subset of $C_\kappa(Z)$. Also for each $n \in \mathbf{N}$, let $F_n^* = F_n \cup \{e\}$, where e is the constant function mapping Z onto 1. Let

$$F_n' = \{rf_1 \cdots f_k | k \in \mathbf{N} \text{ with } k \leqslant n, r \text{ is a real number with } |r| \leqslant 1, \text{and } f_1, \ldots, f_k \in F_1^* \cup \cdots \cup F_n^*\},$$

and let

$$F_n'' = \{f_1 + \cdots + f_k | k \in \mathbf{N} \text{ with } k \leqslant n \text{ and } f_1, \ldots, f_k \in F_n'\}.$$

Finally define $F = \bigcup_{n \in \mathbf{N}} F_n''$.

First we wish to establish that F is a subalgebra of $C_\kappa(Z)$. Let $f, g \in F$; say $f \in F_m''$ and $g \in F_n''$. Then $f = f_1 + \cdots + f_k$, $k \leqslant m$, and $g = g_1 + \cdots + g_l$, $l \leqslant n$, where each $f_i \in F_m'$ and each $g_i \in F_n'$. Each f_i and each g_i is a product of $m + n$ or fewer elements of $F_1^* \cup \cdots \cup F_{m+n}^*$, so that $f + g = f_1 + \cdots + f_k + g_1 + \cdots + g_l$ is in F_{m+n}''. A similar argument will show that $f \cdot g \in F_{mn}''$. Finally, let $t \in \mathbf{R}$ ($\mathbf{R}$ is the space of real numbers). To see that $tf \in F$, let $M \in \mathbf{N}$ such that $|t| \leqslant M$. Now Mf must be in F_{Mm}'' because of what was shown above, so that $tf = (t/M) \cdot Mf$ must also be in F_{Mm}''. Therefore F is a subalgebra of $C_\kappa(Z)$.

Next, if we establish that F separates points, then the Stone-Weierstrass Theorem will tell us that F is dense in $C_\kappa(Z)$. So let $x, y \in Z$ with $x \neq y$. Then there is an $n \in \mathbf{N}$ such that $N_{1/n}(x) \cap N_{1/n}(y) = \varnothing$, where $N_{1/n}(x)$ denotes the open $(1/n)$-ball in Z centered at x. Since F_n is a partition of unity, there is some $f \in F_n$ such that $f(x) \neq 0$. Now there is some $z \in Z$ such that f is supported on $N_{1/n}(z)$. Hence $y \notin N_{1/n}(z)$, so that $f(y) = 0$. Therefore F separates points and is thus dense in $C_\kappa(Z)$.

If we can show that each F_n'' is equicontinuous, then since it is pointwise bounded it would have compact closure in $C_\kappa(Z)$ by the Ascoli Theorem. Therefore $C_\kappa(Z)$ would be almost σ-compact as desired. So let $n \in \mathbf{N}$ be fixed, and let $\varepsilon > 0$ and $x \in Z$. For each i between 1 and n, there exists an open neighborhood U_i of x in Z such that U_i intersects only $B_{i1}, \ldots, B_{ik_i}$ of $\mathfrak{B}_i$. Let $f_{i1}, \ldots, f_{ik_i}$ be the members of F_n supported on $B_{i1}, \ldots, B_{ik_i}$, respectively. For each F_{ij}, there exists a $\delta_{ij} > 0$ such that

$$f_{ij}\left(N_{\delta_{ij}}(x)\right) \subseteq N_{\varepsilon/n^2}\left(f_{ij}(x)\right) \quad \text{and} \quad N_{\delta_{ij}}(x) \subseteq U_i.$$

Define

$$\delta = \min\left\{\delta_{ij} \mid 1 \leqslant i \leqslant n \text{ and } 1 \leqslant j \leqslant k_i\right\}.$$

To see that $g(N_\delta(x)) \subseteq N_\varepsilon(g(x))$ for every $g \in F_n''$, let $g \in F_n''$. Then $g = g_1 + \cdots + g_k$, $k \leqslant n$, where each $g_i \in F_n'$. We may assume without loss of generality that none of the g_i are constant on $N_\delta(x)$. Now each $g_i = r_i g_{i1} \cdots g_{il_i}$, $l_i \leqslant n$, where $|r_i| \leqslant 1$ and each $g_{ij} \in F_1^* \cup \cdots \cup F_n^*$. Here again we may assume that the g_{ij} are not constant on $N_\delta(x)$. Then each g_{ij} is a member of $\{f_{ij} \mid 1 \leqslant i \leqslant n \text{ and } 1 \leqslant j \leqslant k_i\}$, so that

$$g_{ij}\left(N_\delta(x)\right) \subseteq N_{\varepsilon/n^2}\left(g_{ij}(x)\right).$$

But then

$$g_i\left(N_\delta(x)\right) \subseteq N_{\varepsilon/n}\left(g_i(x)\right),$$

and therefore

$$g\left(N_\delta(x)\right) \subseteq N_\varepsilon\left(g(x)\right),$$

as desired. Then F_n'' is equicontinuous, and $C_\kappa(X)$ must be almost σ-compact. □

Now we take $C_\kappa(X)$ to be almost σ-compact and show that under certain mild restrictions, X must be submetrizable.

THEOREM 5. *Let X be a Urysohn k-space. If $C_\kappa(X)$ is almost σ-compact, then X is submetrizable.*

PROOF. Let $F = \bigcup_{n \in \mathbf{N}} F_n$ be a dense subset of $C_\kappa(X)$, where each F_n is compact. For each $f \in C_\kappa(X)$, let R_f be a copy of $\mathbf{R}$, and let $R_n = \prod_{f \in F_n} R_f$. Now for each $n \in \mathbf{N}$, define the function $\varphi_n\colon X \to R_n$ by $\varphi_n(x) = \langle f(x) \rangle_{f \in F_n}$ for each $x \in X$. Suppose temporarily that $n \in \mathbf{N}$ is fixed. If S_f is a subset of R_f for each $f \in F_n$, the expression $\langle S_f \rangle_{f \in F_n}$ will denote the set of elements of R_n whose fth component is in S_f.

We wish to define a base for a uniformity of $\varphi_n(X)$. For each $m \in \mathbf{N}$, define

$$\mu_m = \left[\varphi_n(X) \times \varphi_n(X)\right] \cap \left[\bigcup_{x \in X} \left(\left\langle N_{1/m}(f(x))\right\rangle_{f \in F_n} \times \left\langle N_{1/m}(f(x))\right\rangle_{f \in F_n}\right)\right].$$

Clearly each μ_m is symmetric and contains the diagonal in $\varphi_n(X) \times \varphi_n(X)$. Let $m \in \mathbf{N}$; we wish to establish that there is some $k \in \mathbf{N}$ such that $\mu_k \circ \mu_k \subseteq \mu_m$. Our candidate is $k = 2m$. So let $(\varphi_n(x), \varphi_n(z)) \in \mu_k \circ \mu_k$. Then there is a $y \in X$ such that $(\varphi_n(x), \varphi_n(y)) \in \mu_k$ and $(\varphi_n(y), \varphi_n(z)) \in \mu_k$. There are x_1, $x_2 \in X$ such that for every $f \in F_n$, both $f(x), f(y) \in N_{1/k}(f(x_1))$ and $f(y)$, $f(z) \in N_{1/k}(f(x_2))$. Then for every $f \in F_n$, $f(x) \in N_{1/m}(f(y))$ and $f(z) \in N_{1/m}(f(y))$, so that $(\varphi_n(x), \varphi_n(z)) \in \mu_m$. Therefore $\{\mu_m | m \in \mathbf{N}\}$ is a base for some uniformity $\mathfrak{U}_n$ on $\varphi_n(X)$. Define Z_n to be $\varphi_n(X)$ with the topology induced by $\mathfrak{U}_n$. (Note that Z_n has a larger topology than $\varphi_n(X)$ considered as a subspace of R_n.)

We now need to establish that $\varphi_n: X \to Z_n$ is continuous. Let $m \in \mathbf{N}$, and let $x \in X$. We must find a neighborhood U of x in X such that $\varphi_n(U) \subseteq \mu_m[\varphi_n(x)]$. Since X is a k-space and F_n is compact, then by the Ascoli Theorem, F_n is equicontinuous. So there is a neighborhood U of x in X such that $f(U) \subseteq N_{1/m}(f(x))$ for every $f \in F_n$. But this is precisely what is needed for $\varphi_n(U) \subseteq \mu_m[\varphi_n(x)]$, so that $\varphi_n: X \to Z_n$ must be continuous.

Now we shall let n vary. Notice that each Z_n is metrizable since the uniformity has a countable base. Therefore the product $Z = \prod_{n \in \mathbf{N}} Z_n$ is metrizable. Define $\varphi: X \to Z$ by $\varphi(x) = \langle \varphi_n(x) \rangle_{n \in \mathbf{N}}$ for each $x \in X$. Now φ is continuous since each φ_n is continuous. It remains only to show that φ is injective. Let $x, y \in X$ with $x \neq y$. Then since X is Urysohn, there exists an $f \in C_\kappa(X)$ with $f(x) \neq f(y)$. Let U and V be disjoint open subsets of $\mathbf{R}$ containing $f(x)$ and $f(y)$, respectively. So the basic open subset $W \equiv [\{x\}, U] \cap [\{y\}, V]$ in $C_\kappa(X)$ is nonempty. Then since F is dense in $C_\kappa(X)$, there is an $n \in \mathbf{N}$ and a $g \in F_n$ such that $g \in W$. Since $g(x) \neq g(y)$, then $\varphi_n(x) \neq \varphi_n(y)$. But this means that $\varphi(x) \neq \varphi(y)$, so that φ is injective as desired. Therefore X must be submetrizable. $\square$

Theorem 4 can be extended to spaces of continuous functions mapping into absolute retracts for metric spaces instead of mapping into $\mathbf{R}$. Also Theorem 5 can be extended to spaces of continuous functions mapping into spaces containing arcs as retracts.

We now give an example showing that the k-space hypothesis in Theorem 5 cannot be omitted. Let X be $\mathbf{R}$ with the following topology. Each $\{t\}$ is open for $t \neq 0$, and the open sets containing 0 are the sets containing 0 which have countable complements. This space is called Fortissimo space in [**5**]. It is a regular Lindelöf space, but is not a k-space. Corson showed in [**1**] that $C_\kappa(X)$ is almost σ-compact, but is not separable. Therefore since $C_\kappa(X)$ is not separable and since the cardinality of X is $2^{\aleph_0}$, then X could not be submetrizable because of Vidossich's Theorem in [**6**] (see also Theorem 3 in [**4**]).

Finally we conclude by observing a consequence of Theorems 1 and 4. First note that if a space is submetrizable, then it is almost σ-compact if and only if it is separable. This is because compact submetrizable spaces are metrizable and hence separable.

THEOREM 6. *Let X be a completely regular space. Then X is separable and submetrizable if and only if $C_\kappa(X)$ is separable and submetrizable.*

PROOF. Necessity follows from Theorem 1(c) and Theorem 4. Sufficiency follows from Theorems 1(a) and (c). □

Theorem 6 is in fact true for any topology on the function space which is larger than or equal to the topology of pointwise convergence and smaller than or equal to the compact-open topology.

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