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ANOTHER APPROXIMATION THEORETIC CHARACTERIZATION OF INNER PRODUCT SPACES

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ABSTRACT. A normed space E is an inner product space if and only if for every 2-dimensional subspace V and every segment $I \subset V$, the corresponding metric projections satisfy the commutative property $P_I P_V = P_V P_I$.

For a subset A of a normed linear space E, we denote by P_A the *metric projection* on A, i.e. the set-valued mapping which corresponds to each $x \in E$ the (possibly empty) set of its best approximations in A: $P_A x = \{y \in A; \|x - y\| = d(x, A)\}$. A is called *proximinal* if $P_A x$ is nonempty for every $x \in E$. If E is a Hilbert space and A is a closed subspace of E, then P_A is just the (single-valued) orthogonal projection onto A.

There are several known characterizations of inner product spaces which can be stated in terms of the metric projections. See e.g. **[3]**, **[5]**, **[8]**, and **[9]**. We shall consider three other such conditions below. For all of these three characterizing conditions, the necessity part is immediate. The weakest condition (hence strongest characterization) is due to Lorch **[7]** (cf. also Day **[1**, p. 152]):

(L) *If* $\|x\| = \|y\| = 1$ *and* $A = \{\beta x + y/\beta;\ \beta \neq 0$ *real*$\}$, *then* $x + y \in P_A 0$.

The next condition, due to Gurari and Sozonov **[4]**, is:

(GS) *If* $\|x\| = \|y\| = 1$ *and A is the segment* $[x, y] = \{\alpha x + (1 - \alpha)y;\ 0 \leqslant \alpha \leqslant 1\}$, *then* $(x + y)/2 \in P_A 0$.

The third condition (under the assumption $\dim E \geqslant 3$) is due to Joichi **[6]**:

(J) *If V is a 2-dimensional subspace of E, $u \in E$ with $0 \in P_V u$ and $A = S_0$ is the "unit circle" in V:* $S_0 = \{v \in V: \|v\| = 1\}$, *then* $P_A u = A$.

We first give an easy proof that (J) $\Rightarrow$ (GS) $\Rightarrow$ (L), and hence this yields an alternate approach to the more involved sufficiency proofs as given in **[4]** and **[6]**.

(J) $\Rightarrow$ (GS): We may assume $\dim E = 3$. If (GS) fails, then there exists x, y in E with $\|x\| = \|y\| = 1$ and $0 \leqslant \gamma < \frac{1}{2}$ such that the element $z = \gamma x + (1 - \gamma)y$ satisfies $\|z\| = \min_{0 \leqslant \lambda \leqslant 1/2} \|\lambda x + (1 - \lambda)y\| < \|\frac{1}{2}(x + y)\|$. Extend the segment $[x, y]$ to a supporting hyperplane $V + z$ to the ball $\{w \in E:$

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$\|w\| \leqslant \|z\|\}$ at z. Let $u = -z/\|x - z\|$ and $v = (x - z)/\|x - z\|$. Then $0 \in P_V u$ and $\pm v \in S_0$, but

$$\|u - v\| = \frac{\|x\|}{\|x - z\|} \neq \frac{\|2z - x\|}{\|x - z\|} = \|u + v\|$$

(since if $\|x\| = \|2z - x\| = \|y\|$, then the three distinct collinear points x, y, and $2z - x$ have the same norm so all the points in $[x, y]$ must have the same norm, a contradiction to $\|z\| < \|\frac{1}{2}(x + y)\|$). Thus (J) fails.

(GS) $\Rightarrow$ (L): Suppose (GS) holds. If $\|x\| = \|y\| = 1$, $\beta \neq 0$, and $\gamma = \beta/2 + 1/2\beta$, then $|\gamma| \geqslant 1$. Hence

$$\left\| \beta x + \frac{1}{\beta} y \right\| \geqslant 2 \left\| \frac{\beta}{2\gamma} x + \frac{1}{2\beta\gamma} y \right\| \geqslant 2 \left\| \frac{1}{2}(x + y) \right\| = \|x + y\|$$

(since $\beta/2\gamma + 1/2\beta\gamma = 1$). Hence (L) holds.

The characterization we add here is by a commutativity property of the metric projection.

THEOREM. *The following are equivalent for a normed space E with* $\dim E \geqslant 3$:

(1) *E is an inner product space;*

(2) *For every proximinal subspace V and any nonempty $A \subset V$, $P_A \circ P_V = P_A$ $(= P_V \circ P_A)$;*

(3) *The same as* (2) *with V 2-dimensional and A a segment.*

PROOF. (1) $\Rightarrow$ (2). This is immediate from the orthogonality of $x - P_V x$ to V: if $x \in E$, $y = P_V x$, $z \in P_A y$, and $a \in A$, then

$$\|x - z\|^2 = \|x - y\|^2 + \|y - z\|^2 \leqslant \|x - y\|^2 + \|y - a\|^2 = \|x - a\|^2$$

and equality holds if and only if $\|y - a\| = \|y - z\|$, i.e. $a \in P_A y$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Since a normed space E is an inner product space iff each 2-dimensional subspace is, we may assume $\dim E = 3$.

We show first that E is strictly convex. If not, we can find a 2-dimensional subspace V and $x \in E$ such that $P_V x$ is not a singleton. We may assume also that 0 is a boundary point of $P_V x$ in V. Let K be the cone $\{\lambda P_V x: \lambda \geqslant 0\}$. K is contained in a halfspace, therefore $K \neq V$ and we can find $v \in V$, $\|v\| = 1$, with $0 < d(v, K) < \frac{1}{2}$ and $\lambda > 0$, $y \in P_V x$ with $\|\lambda y - v\| < \frac{1}{2}$. Let $A = [0, v]$. Since $v \notin K$, $A \cap P_V x = \{0\}$ and $P_A x = 0$. On the other hand, if $0 \in P_A y$, then $\|y\| \leqslant \|y - \gamma v\|$ for all $\gamma \in [0, 1]$, hence for all $\gamma \geqslant 0$. It follows that $\|\lambda y\| \leqslant \|\lambda y - \lambda\gamma v\|$ for all $\gamma \geqslant 0$. In particular, $\|\lambda y\| \leqslant \|\lambda y - v\| < \frac{1}{2}$ while $\|\lambda v\| \geqslant \|v\| - \|\lambda y - v\| > \frac{1}{2}$, a contradiction. Thus E is strictly convex.

We shall show that for every 2-dimensional subspace V of E and $x \in E$ with $P_V x = 0$, the nonempty intersections of "spheres" around x with V are multiples of the unit circle in V, i.e. if $y_0 \in V$ is arbitrary, then $S_1 = \{v \in V: \|x - v\| = \|x - y_0\|\}$ coincides with $\|y_0\| S_0 = \{v \in V: \|v\| = \|y_0\|\}$. By (J), this guarantees that E is an inner product space.

The idea of the proof is to show that S_1 is a curve "parallel" to S_0, i.e. that every line supporting one of them is parallel to a line supporting the other at the corresponding point. If not, we can find a line segment $[y, z]$, $y \in S_1$, such that $(y, z]$ is contained in one of the domains bounded by S_1 and $\|y\| S_0$ and disjoint to the other, so that y is exactly one of the points $P_{[y,z]}x$ or $P_{[y,z]}0$. It is very natural to conclude from this that S_1 and S_0 are proportional, but the formal proof we have uses differentiability properties of convex functionals.

Let ρ_i be the Minkowski functional in V of the convex hull of S_i, i.e. $\rho_0(v) = \|v\|$, $\rho_1(v) = 1$ if $v \in S_1$, and ρ_1 is positively homogeneous. Since E is strictly convex, ρ_1 is well defined. The ρ_i are convex and therefore $\Delta_i(y, w; t) = t^{-1}[\rho_i(y + tw) - \rho_i(y)]$ is a nondecreasing function of $t > 0$ and $\tau_i(y, w) = \lim_{t\to 0^+}\Delta_i(y, w; t)$ exists for every y, w in V (see [2, p. 446]). If $\gamma > 0$, then

$$\tau_i(\gamma y, w) = \lim_{t\to 0^+} \Delta_i(\gamma y, w; t) = \lim_{t\to 0^+} \Delta_i\left(y, w; \frac{t}{\gamma}\right) = \tau_i(y, w).$$

Fix $y \in S_1$ and $w \in V$ which is not in span$\{y\}$. Let $\gamma_1 = 1$ and $\gamma_0 = 1/\|y\|$, so that $\gamma_i y \in S_i$ and hence $\rho_i(\gamma_i y) = 1$ for $i = 0, 1$. Denote $\tau_i = \tau_i(y, w)$. Consider the lines

$$l_i(\beta) = l_i(y, w; \beta) = (1 - \beta\tau_i)\gamma_i y + \beta w, \qquad \beta \text{ real.}$$

If $\beta > 0$ is small enough so that $\beta\tau_i < 1$, we have

$$\rho_i(l_i(\beta)) - 1 = \beta\left[\Delta_i\left(\gamma_i y, w; \frac{\beta}{1 - \beta\tau_i}\right) - \tau_i\right] \geqslant 0.$$

If $S_i(\beta) = l_i(\beta)/\rho_i(l_i(\beta))$ (the radial projection of $l_i(\beta)$ onto S_i), then

$$\frac{1}{\beta}\,\rho_i\big[l_i(\beta) - S_i(\beta)\big] = \Delta_i\left(\gamma_i y, w; \frac{\beta}{1 - \beta\tau_i}\right) - \tau_i \to 0 \quad \text{as } \beta \to 0^+,$$

i.e. $\rho_i(l_i(\beta)) = 1 + o(\beta)$ as $\beta \to 0^+$. (This means that $l_i(\beta)$ is "tangent" to S_i at $\gamma_i y$ from the $\beta > 0$ direction.)

If $\gamma_i\tau_i < \gamma_j\tau_j$, then for some $\varepsilon > 0$, $\gamma_i\Delta_i(\gamma_j y, w; t) < \gamma_j\tau_j$ for all $0 < t \leqslant \varepsilon$. Let

$$z = \frac{1}{\gamma_j}\, l_j\left(\frac{\varepsilon}{1 + \varepsilon\tau_j}\right) \quad \text{and} \quad A = [y, z].$$

If $u \in (y, z]$, i.e. $u = l_j(\beta)/\gamma_j$ for $0 < \beta \leqslant \varepsilon/(1 + \varepsilon\tau_j)$, then $0 < \beta/(1 - \beta\tau_j) \leqslant \varepsilon$ and

$$\begin{aligned}\rho_k(u) - \rho_k(y) &= \rho_k\big[(1 - \beta\tau_j)y + \beta w/\gamma_j\big] - \rho_k(y)\\ &= \frac{\beta}{\gamma_k\gamma_j}\big[\gamma_k\Delta_k(\gamma_j y, w; \beta/(1 - \beta\tau_j)) - \gamma_j\tau_j\big]\end{aligned}$$

is negative for $k = i$ and nonnegative for $k = j$, i.e. $\rho_i(u) < \rho_i(y)$ and $\rho_j(u) \geqslant \rho_j(y)$ for all $u \in (y, z]$. In the case $i = 0$, we thus have $\|u\| < \|y\|$

and $\|x - u\| \geqslant \|x - y\|$ for all $u \in (y, z]$ so that $P_A x = y \neq P_A 0 = P_A P_V x$, which contradicts (3). While in the case $i = 1$, we have $\|u\| \geqslant \|y\|$ and $\|x - u\| < \|x - y\|$ for all $u \in (y, z]$ so that $P_A P_V x = P_A 0 = y \neq P_A x$, again contradicting (3). Thus we must have $\gamma_i \tau_i = \gamma_j \tau_j$, i.e. $\tau_0 = \|y\| \tau_1$ and therefore $l_1(\beta) = \|y\| l_0(\beta / \|y\|)$. Since $\rho_1(l_1(\beta)) = 1 + o(\beta)$ and $\rho_0(l_0(\beta/\|y\|)) = 1 + o(\beta)$, we have $\rho_0(l_1(\beta)) = \|y\| + o(\beta)$ as $\beta \to 0^+$. Therefore, the function

$$\psi(\beta) = \frac{\rho_0(l_1(\beta))}{\rho_1(l_1(\beta))}, \qquad \beta \geqslant 0,$$

(which represents the ratio between the radially corresponding points on S_0 and S_1 in the one-sided neighborhood of the y-direction determined by w) satisfies

$$\psi'_+(0) = \lim_{\beta \to 0^+} \frac{\psi(\beta) - \|y\|}{\beta} = 0.$$

But the same applies to the derivative of the ratio from the other direction, so that the two-sided derivative of this ratio at y is 0. Since y was any point on S_1, the ratio is the constant $\|y_0\|$, i.e. $S_1 = \|y_0\| S_0$. $\square$

Added in proof. The proof of the implication (3) $\Rightarrow$ (1) can be substantially shortened by eliminating everything that comes after the paragraph which shows E is strictly convex, and substituting the following in its place. Recall Hirschfeld's characterization [3]: If E is a strictly convex non-Euclidean 3-dimensional space, there is a one-dimensional subspace L with P_L nonlinear. Let V be any 2-dimensional subspace of E containing L. Since metric projections onto maximal subspaces are linear, P_V and $P_L P_V$ are linear, which shows that $P_L \neq P_L P_V$. Take any x with $P_L x \neq P_L P_V x$ and let $A = [P_L x, P_L P_V x] \subset L$. Then $P_V P_A x = P_A x = P_L x \neq P_L P_V x = P_A P_V x$.

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