PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 71, Number 2, September 1978

CERTAIN IDEMPOTENTS LYING IN THE CENTRALIZER OF THE GROUP OF UNITS

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ABSTRACT. Let S be a compact connected monoid of dimension n having G as a connected group of units. Let B be a closed subgroup outside of the minimal ideal. The maximum dimension possible for the product BG is $n - 1$. If this maximum is attained by BG and GB and both are Lie groups then B meets the centralizer of G.

Let S be a compact connected monoid whose group of units G is connected. Our purpose here is to show that certain subgroups of S must meet, and in some cases lie in $Z(G, S)$, the centralizer of the group of units. These results stem in part from the manner in which an (algebraically) irreducible submonoid is embedded. For example, it is shown [1] that an irreducible submonoid cannot have dimension at the unit exceeding $\dim S - \dim G - 1$. There, the important item was the structure of the product GB where B was a subgroup of S.

If the closed subgroup B lies outside of the minimal ideal the maximum dimension possible for GB is $\dim S - 1$. It is this case which we shall pursue. If S is finite dimensional with G and B both Lie group then the condition

$$\dim BG = \dim GB = \dim S - 1$$

will cause B to meet $Z(G, S)$. If G happens to be semisimple, $\dim S - 2$ will suffice in place of $\dim S - 1$.

Here, we shall be concerned with the products of the form GBG. We show that if B is normal in its maximal subgroup then GBG is the space of a fiber bundle where the fiber is the homogeneous BG and the base is a quotient of G.

Suppose now that S and G are as above and that e is an idempotent belonging to the compact subgroup B. Further, suppose that B is disjoint from the minimal ideal as well as G.

In order to show that a given e is in $Z(G, S)$, we must show that $Ge \subseteq eS$ and $eG \subseteq Se$. This is equivelent to $Ge \cup eG \subseteq H_e$–the maximal subgroup of e. Another equivalent formulation is that $Ge \subseteq BG$ and $eG \subseteq GB$. We first concentrate upon the condition that $Ge \subseteq BG$. The following definition is appropriate:

DEFINITION. Let $Y = Y(B, G) = \{g | g \in G, gBG = BG\}$. Clearly, Y is a

Received by the editors July 17, 1975 and, in revised form, January 26, 1978.
AMS (MOS) subject classifications (1970). Primary 22A15; Secondary 22C05, 22E20.
[1]With the support of the National Science Foundation MPS 74-07279.

closed subgroup of G. The elements of Y may be described in a number of ways: (1) $g \in Y$; (2) $gBG \subseteq BG$; (3) $gB \subseteq BG$; (4) $ge \in \bigcap\{BGb\colon b \in B\}$. Consider, however, the following condition

$$gBG \cap BG \neq \varnothing.$$

There is no apparent reason to suppose that this condition will place g in Y. A moment's reflection, however, indicates that this condition is what is needed so that the sets $\{gBG\}$, $g \in G$, do form a decomposition.

DEFINITION. We shall say that B is meshed with G, (on the left), if $gBG \cap BG \neq \varnothing$ implies $g \in Y$, i.e. $gBG = BG$.

The appropriateness of this notion is shown in the following proposition which is fundamental.

PROPOSITION 1. *Let G be the group of units of a compact monoid and let B be a compact subgroup outside of G and the minimal ideal. If B is meshed with G then there is an open surjection $p\colon GBG \to G/Y$ given by $p(gBG) = gY$ with $p^{-1}(gY) = gBG$. Thus, $(GBG, p, G/Y)$ is a fiber space. If* dim *G/Y is finite then $q\colon G \to G/Y$ admits a local cross-section and the fiber space $p\colon GBG \to G/Y$ is a locally trivial fiber bundle with fiber BG. The associated bundle is $G \to G/Y$.*

PROOF. Since B is meshed with G it is clear that p is well defined and that $p^{-1}(gY) = gBG$. The fibers $[gGB]$ are clearly all homeomorphic with BG. Let 0 be an open set in GBG and let V and W be such that $V \times W$ is the inverse image of 0 under the multiplication $G \times BG \to GBG$. Now $p(VBG) = VY$ which is certainly open in G/Y since $G \to G/Y$ is open. But $p(0) = p(VBG)$. Thus p is open. Suppose now that G/Y is of finite dimension. We know from [5] that local cross-sections exist. Thus let U be an open set in G/Y and $s\colon U \to G$ a continuous map with $qs(u) = u$ for $u \in U$. Define the homeomorphisms h and f by

$$h\colon U \times Y \to q^{-1}(U), \qquad h(u,y) = s(u)\cdot y,$$
$$f\colon U \times BG \to p^{-1}(U), \qquad f(u,x) = s(u)\cdot x.$$

Then the following diagrams commute. The first indicates the bundle structure of $q\colon G \to G/Y$, which is well known and the second that of $p\colon GBG \to G/Y$.

$$\begin{tikzcd} U \times Y \arrow[rr, "h"] \arrow[dr] & & q^{-1}(U) \arrow[dl] \\ & U & \end{tikzcd} \qquad \begin{tikzcd} U \times BG \arrow[rr, "f"] \arrow[dr] & & p^{-1}(U) \arrow[dl] \\ & U & \end{tikzcd}$$

The unmarked arrows being projections and p and q appropriately cut down. Now let $w \in U \cap U_0$ with U_0 and s_0 playing the role of U and s. Combining the transition functions

$$f_0^{-1}f(w,x) = \big(w, s_0^{-1}(w)s(w)x\big).$$

Hence the associated transformations of the fiber are of the form $x \to x_0^{-1}(w)s(w)x$ with $s_0^{-1}(w)s(w) \in Y$. These are the transformations of the principal bundle $g\colon G \to G/Y$. By definition, the associated bundle has G/Y as base and Y as fiber. We see that $q\colon G \to G/Y$ is indeed the associated bundle. The condition that B be meshed with G is not unreasonable. We see this now.

LEMMA 1. *If $eGe \cap H_e$ lies in the normalizer of B in H_e then B is meshed with G. In particular, if C is the component of e in H_e then C and G are meshed.*

PROOF. Suppose $gBG \cap BG \neq \varnothing$. Let $x_1, x_2 \in B, y_1, y_2 \in G$ with $gx_1y_1 = x_2y_2$. Clearly $gx_1G = x_2G$ and $Bgx_1G = BG$. We may then write

$$Bgx_1G = Begex_1G = egeBx_1G = egBG = BG.$$

But gBG lies entirely in the R-class of e since gBG meets BG. But then $egBG = gBG$ so that $gBG = BG$.

When studying the fiber space $GBG \to G/Y$ we are often concerned with the conclusion that G/Y be nondegenerate. In the important case in which B is a group component we have the following:

LEMMA 2. *Let S be a compact monoid whose group of units G is connected. Let C be the component of e in H_e, where $e^2 = e$. If $GC \neq C$ then $G/Y \neq \varnothing$.*

PROOF. First, GC lies in the $\mathfrak{L}$-class of e. We cannot have $GC \subseteq H_e$ since this would imply $GC = C$. Thus, part of GC lies outside of the $\mathfrak{R}$-class of e. But the last contains CG. Thus $GC \neq CG$ so that $GCG \neq CG$. Then, by definition, $G/Y \neq 0$ where $Y = \{g \mid gCG \subseteq CG\}$.

Let us note the following fact: If G is a group of units and B is a subgroup then $BG = B$ implies GB is a left simple semigroup and hence a left group. In effect, $GBgb = GBb = GB$.

In order to continue our study of GBG we recall some material from **[1]**.

DEFINITION. Let $W = \{g \mid gB = B, g \in B\}$.

Observe that $g \in W$ if and only if $gB \subseteq B$ if and only if $ge \in B$.

LEMMA 3. *The group $G \times B$ acts on Se via $(b, g) \cdot x = gxb^{-1}$. Moreover, the map $g \to (g, ge)$ takes W isomorphically onto the isotropy group $(G \times B)_e$ at e.*

PROOF. Observe that $w \to we$ is a homomorphism. Furthermore, $(g, b) \in (G \times B)_e$ if and only if $b = ge$.

LEMMA 4. *Multiplication, $(g, b) \to g \cdot b$ is a principal fibration $G \times B \to GB$, with fiber W. This fibration is the quotient map of a homogeneous space.*

PROOF. Since GB is the orbit of $G \times B$ at e the map is equivalent to the quotient map defined by $(G \times B)_e$.

For our purposes it is necessary to determine dim GBG. This is now given:

PROPOSITION 2. *Let B and G, given as above, be meshed. If G and B are finite dimensional then* $\dim GBG = \dim BG + \dim G/Y$.

PROOF. From the preceding, BG is locally the product of a zero dimensional compact set and a cell of dimension equal to dim BG. Likewise G/Y has its dimensions determined by a cell of the same dimension as G/Y. Since GBG is locally a product of open sets from BG and G/Y, we have the desired conclusion.

Let us note that if G is connected and $\dim G/Y = 0$, which is the same thing as $\dim BG = \dim GBG$, then $Ge \subseteq BG$, by the definition of Y. Thus, the larger the dimension of BG the more likely our desired conclusion.

Now let S be a compact connected finite dimensional monoid having a group of units G and a compact subgroup B outside of the minimal ideal. Then the maximum dimension possible for BG is $\dim S - 1$. One can see this in two different ways. Using the action of $B \times G$ upon BS one can cite [2] directly. On the other hand, BG is a homogeneous space of $B \times G$ and so has nonzero element in $H^k(BG)$ where $k = \dim BG$. This is ruled out by [7].

DEFINITION. With B and G as above, not necessarily meshed however, we shall say that BG is of maximal dimension if $\dim BG = n - 1$.

LEMMA 5. *If BG is of maximal dimension then B is meshed with G.*

PROOF. If C denotes the identity component of e in the group H_e then clearly $\dim CG = \dim BG$. However, from the remarks before, $C \times G$ yields the homogeneous space CG and $B \times G$ yields the homogeneous space BG. But a compact connected homogeneous space cannot contain properly another compact homogeneous space of the same finite dimension.

Suppose now that S, G, B are as above with $\dim S = n$ and BG of maximal dimension. Then we either have $\dim GBG = n - 1 = \dim BG$ or we have the (unlikely) situation $\dim GBG = n$. In view of a number of considerations to follow, an appropriate way of eliminating the second possibility is that GBG have nonzero cohomology in its top dimension, i.e. $H^{\dim GBG}(GBG) \neq 0$. This is due to a well-known result of Wallace. See [7] or [4].

Thus, we may state the following.

LEMMA 6. *Let S be a compact connected monoid of finite dimension having a connected group of units G. Let B be a compact connected subgroup (outside of the minimal ideal) such that GB and BG are of dimension* $\dim S - 1$. *Then if $H^k(GBG) \neq 0$, where $k =$* $\dim GBG$, *the subgroup B meets $Z(G, S)$.*

Observe now that if B is meshed with G and both are Lie groups then GBG is a manifold. In effect GBG is fibered over the manifold G/Y by the manifold BG. Thus, in this case, $H^{\dim(GBG)}(GBG) \neq 0$.

COROLLARY 1. *Let S be a compact connected monoid of finite dimension whose group of units G, is a connected Lie group. Suppose e is an idempotent outside of the minimal ideal and that C the group component of e is also a Lie group. If* $\dim GB = \dim BG = \dim S - 1$ *then e belongs to $Z(G, S)$.*

If G is semisimple then

$$\dim BG \geqslant \dim S - 2 \leqslant \dim GB$$

imply that e belongs to $Z(G, B)$.

PROOF. For the second claim, recall that a compact connected semisimple Lie group cannot yield a homogeneous space of dimension one. Thus, $\dim G/Y$ is at least two unless $GC = C$. We would then have the manifold GCG of dimension

$$\dim G/Y + \dim CG \geqslant 2 + (\dim S - 2) = \dim S,$$

which is impossible.

COROLLARY 2. *Let S be a compact connected monoid of dimension n, having a connected group of units G. If e is an idempotent outside of G and the minimal ideal such that* $\dim H_e = n - 1$ *then* $e \in Z(G, S)$.

PROOF. We note that GC is a homogeneous space of $G \times C$. Were we to have C properly contained in GC we must have $\dim GC = n$, since a homogeneous space which is compact and connected may not contain another of the same finite dimension. Thus $GC = C$ and, in the same way, $CG = C$.

COROLLARY 3. *Let S be a compact connected finite dimensional monoid whose group of units G is a connected semisimple Lie group. Let e be an idempotent belonging to neither Z(G, S) nor the minimal ideal. If C the maximal compact connected subgroup at e is a Lie group then* $\dim C \leqslant \dim S - 3$.

PROOF. Since $e \notin Z(G, S)$ we cannot have $GC = CG = C$. Say $GC \neq C$. Then consider the fibering $GCG \to G/Y$. Since $GC \subsetneq C$ we know that $GC \subsetneq CG$. Thus the quotient G/Y is nondegenerate. Since G is semisimple $\dim G/Y \geqslant 2$. Now, $\dim GCG = \dim CG + \dim G/Y$. Thus, if $\dim C = \dim S - 2$ we would have $\dim GCG = \dim S$ which, as we know, is impossible. Hence $\dim C \leqslant \dim S - 3$.

COROLLARY 4. *Let S be a compact connected monoid having a group of units G which is a connected Lie group. Let e be an idempotent belonging to neither Z(G, S) nor the minimal ideal. Let C denote the component of e in* H_e *and suppose that C is a Lie group. If either*

(1) $\dim C = \dim S - 2$ *or*

(2) *G is semisimple and* $\dim C = \dim S - 3$,

holds then either GC or CG (not both) coincides with C. Thus, say, GC is a left group with C as maximal subgroup.

PROOF. Consider again $GCG \to G/Y$. Suppose first, $\dim C = \dim S - 2$. Then if both $GC \neq C$ and $CG \neq C$ we would have $\dim GCG = \dim CG + \dim G/Y$. Since $C \subsetneq CG$ we have $\dim CG \geqslant \dim S - 1$. Since $GC \neq C$ we have $GCG \subsetneq CG$ so that $\dim G/Y \geqslant 1$. Thus, $\dim GCG = \dim S$ which is impossible. Now suppose that $\dim C = \dim S - 3$ with G semisimple. In this case if $GC \neq C$ and $CG \neq C$ we have

$$\dim GCG = \dim CG + \dim G/Y \geqslant \dim S - 2 + 2 = \dim S.$$

Again, a contradiction.

EXAMPLE. Let G be the three sphere as a topological group using quaternions of norm one. Following [3] let $|G|$ be the space G with left zero multiplication. Then $L = |G| \times G$ is a left group. Let G act on L by $g(g', g'') = (gg', gg'')$ and $(g', g'')g = (g', g''g)$. Then $L \cup G$ is a compact monoid in which $e = (1, 1)$, where 1 is the unit of G, is not in the centralizer of G. Instead of G one may use some homogeneous space of G say G/K. Again $G \cup \{|G/K|G\}$ is a monoid. We may take $S = G \cup \{|G/K|G\}$ of dimension five with G/K say S^2. Now take the cone over S and get a compact connected monoid of dimension six with e having its component of dimension three $= n - 3$. Here e is not in the centralizer of the group of units. However the set corresponding to GC is again a left group. See [3] for further details.

As the reader has noted, the fibering of GBG by BG over G/Y shows that the total space has nontrivial cohomology in top dimension if B and G are manifolds, i.e. Lie groups. It thus seems reasonable to conjecture that if (E, B, F) is a fibre bundle with space E, base B and fibre F then B and F are compact connected finite dimensional topological groups one could conclude that $H^n(E, Z) \neq 0$ where $n = \dim B + \dim F$. (It is known that $H^{\dim B}(B)$ and $H^{\dim F}(F)$ are nonzero over Z.) Oddly enough such is not the case.

EXAMPLE. *There exists a space E which is a fibre bundle over the circle fibered by the dyadic solenoid with* $H^2(E, Z) = 0$.

Let K be the Klein bottle obtained as usual by identification of the ends of a cylinder $I \times T$ through the diameter. On T the circle this is the map $Z \to Z^{-1}$. Since this identification map is compatible with the squaring map $Z \to Z^2$ on T we may map K onto itself preserving the base and wrapping each fibre twice. This map α induces α^* on $H^2(K, Z)$ which is trivial.

Recall that $H^2(K, Z) = Z_2$. From direct considerations, the kernel of α^* is again Z_2, i.e. all of $H^2(K, Z)$. The inverse limit of a square of such spaces K and maps α will yield a fibre bundle E with base a circle and fibre the dyadic solenoid. However,

$$H^2(E, Z) = 0.$$

We mention that Floyd has constructed a free action of p-adic group on an acyclic space [6].

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