PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 71, Number 2, September 1978

K-THEORY AND K-HOMOLOGY RELATIVE TO A II_∞-FACTOR

IAIN RAEBURN

ABSTRACT. Let X be a compact space and M be a factor of type II_∞ acting on a separable Hilbert space. Let $K_M(X)$ denote the Grothendieck group generated by the semigroup of isomorphism classes of M-vector bundles over X, and, if X is also metric, let $\mathrm{Ext}^M(X)$ denote the group of equivalence classes of extensions of $C(X)$ relative to M. We show that $K_M(X)$ is the direct sum of the even-dimensional Čech cohomology groups of X, and that $\mathrm{Ext}^M(X)$ is the direct product of the odd-dimensional Čech homology groups of X.

Introduction. Recently Brown, Douglas, and Fillmore [**8**] have constructed a generalised homology theory called K-homology, which, in a sense made rigorous in [**8**], is dual to K-theory. Their construction is in terms of extensions of commutative C^*-algebras by the ideal of compact operators on a separable Hilbert space. Fillmore [**12**] and Cho [**9**] have investigated the analogous construction with the compact operators replaced by the closed two-sided ideal generated by the finite projections in a factor of type II_∞. They have constructed (see [**9**]) a generalised homology theory $\{\mathrm{Ext}_n^M\}$ on the category of compact metric spaces, which we shall call K-homology relative to the II_∞-factor M. In [**6**] Breuer has considered a theory of vector bundles relative to M and introduced a functor K_M which has topological properties like those of K-theory. We shall construct a generalised cohomology theory $\{K_M^n\}$ (K-theory relative to M) from Breuer's functor, identify it in terms of the conventional K-functor and show that $K_M(X)$ is the direct sum of the even-dimensional real cohomology of X for any compact space X. Then we shall deduce the corresponding result for Ext^M; namely that $\mathrm{Ext}_1^M(X)$ is the direct product of the odd-dimensional real homology of X. We mention that the results in this note all follow in standard fashion from the recent literature; our goal is merely to point out some interesting consequences of the work of Breuer [**6**] and Cho [**9**]. Along the way we provide a proof of Proposition 2, which has been stated and used by Singer in [**18**].

First we set up some notation. Throughout, all topological spaces will be Hausdorff, and M will be a factor of type II_∞ acting on a separable Hilbert space H. We shall denote by $P_f(M)$ the set of finite projections of M and by dim: $P_f(M) \to \mathbf{R}^+$ the Murray-von Neumann dimension function of M. For

Received by the editors November 21, 1977 and, in revised form, January 25, 1978.
AMS (MOS) subject classifications (1970). Primary 55B20; Secondary 46L10, 47B30, 47C15.

details on such matters, we refer to [**10**]. In addition, we shall write $\mathcal{K}(M)$ for the closed two-sided ideal of M generated by $P_f(M)$, $\mathfrak{A}(M)$ for the quotient algebra $M/\mathcal{K}(M)$ and $\mathcal{F}(M)$ for the set of operators which are Fredholm relative to M (cf. [**5**]). Our terminology as regards K-theory will be that of [**1**]. By a generalised (Čech) cohomology theory on compact pairs, we shall mean a sequence $\{K^n\}$ of contravariant functors which satisfy the three axioms of continuity, excision and exactness (cf. [**20**, §1]). We observe that continuous functors are necessarily homotopy invariant [**20**, Theorem 2.1], so that such theories satisfy the first six of the Eilenberg-Steenrod axioms. We shall need the following lemma.

LEMMA. *If μ: $\{H^n\} \to \{K^n\}$ is a natural transformation between generalised cohomology theories such that μ: $H^n(X) \to K^n(X)$ is an isomorphism for all n when X is a point, then μ is a natural equivalence.*

PROOF. That μ is an equivalence on compact polyhedra follows from the argument of [**19**, Theorem 4.8.10]. But every compact space is the inverse limit of spaces with the homotopy type of compact polyhedra [**19**, Lemma 6.6.7], and so the result holds on the category of compact spaces.

1. Let X be a compact space. Breuer [**6**] introduced the notion of an M-vector bundle over X–namely, a Hilbert space bundle over X whose transition functions take values in M and whose fibres are of the form $E(H)$ for some E belonging to $P_f(M)$. The set $\mathrm{Vect}_M(X)$ of M-isomorphism classes of M-vector bundles over X is a semigroup under direct sum; if f is a continuous map from Y to X, then f induces (via pull-back of bundles) a semigroup homomorphism f^*: $\mathrm{Vect}_M(X) \to \mathrm{Vect}_M(Y)$. If we denote the Grothendieck group of $\mathrm{Vect}_M(X)$ by $K_M(X)$, then K_M is a contravariant functor from compact spaces to abelian groups. If X is a compact space with distinguished base point x_0, and i: $\{x_0\} \to X$ is the inclusion, then we write $\tilde{K}_M(X)$ for the kernel of the map i^*: $K_M(X) \to K_M(\{x_0\})$. Breuer proved that K_M is homotopy invariant, and that $K_M(X)$ is a module over the ring $K(X)$; it is easy to check from the definition [**6**, p. 417] that this module action is natural. The main result of Breuer's article is the periodicity theorem for K_M; namely that for any locally compact space X, $K_M(\mathbf{R}^2 \times X) \cong K_M(X)$, where for Y locally compact $K_M(Y)$ stands for the reduced group $\tilde{K}_M(Y \cup \{\infty\})$ of the one point compactification of Y. This isomorphism is natural since the inverse β_X is defined in terms of the module action.

We define $K_M^{-n}(X) = K_M(\mathbf{R}^n \times X)$ (for $n > 0$) and, inductively, $K_M^n(X) = K_M^{n-2}(X)$ for positive n. If for a compact pair (X, Y) we now set $K_M^n(X, Y) = \tilde{K}_M^n(X/Y)$ (the base point is Y/Y) then $\{K_M^n\}$ is a sequence of contravariant functors from compact pairs to abelian groups.

PROPOSITION 1. *$\{K_M^n\}$ is a generalised cohomology theory on compact pairs.*

PROOF. That $\{K_M^n\}$ satisfies excision is obvious. To verify continuity and exactness we shall use the theorem of Breuer that $K_M(X) \cong [X, \mathcal{F}(M)]$ (see

[**6**, Theorem 1, p. 414]); an inspection of Breuer's construction yields that the isomorphism is natural. Since $\mathcal{F}(M)$ is an open set in the Banach space M ([**5**, II, Corollary 2 to Theorem 1]), $\mathcal{F}(M)$ and its loop spaces $\Omega^n\mathcal{F}(M)$ are ANR's (cf. [**14**, Chapter 1]). It follows from the periodicity theorem that $\pi_n(\mathcal{F}(M)) \cong \pi_n(\Omega^2\mathcal{F}(M))$ for every $n \geqslant 0$, and so $\mathcal{F}(M)$ and $\Omega^2\mathcal{F}(M)$ are homotopy equivalent by [**17**, Theorem 15]. Thus $\{K_M^n\}$ is given by a spectrum, and so by [**21**, §5] satisfies the exactness axiom on finite complexes. We can deduce that K_M is continuous from the fact that $\mathcal{F}(M)$ is an ANR, and the result follows. □

If X is a compact space, $r \in \mathbf{R}^+$ and $E \in P_f(M)$ satisfies $\dim E = r$, then, as in the construction of the module action [**6**, p. 417], there is a map λ_r: $\mathrm{Vect}(X) \to \mathrm{Vect}_M(X)$ given by $\lambda_r(a) = a \otimes (X \times E(H))$. Thus there is a pairing $(a, r) \to \lambda_r(a)$: $\mathrm{Vect}(X) \otimes \mathbf{R}^+ \to \mathrm{Vect}_M(X)$ which induces a natural transformation λ: $K(\cdot) \otimes_{\mathbf{Z}} \mathbf{R} \to K_M(\cdot)$. We observe that λ: $K(X) \otimes \mathbf{R} \to K_M(X)$ is an isomorphism when X is a one point space.

PROPOSITION 2 (SINGER). *The functors $K(\cdot) \otimes_{\mathbf{Z}} \mathbf{R}$ and $K_M(\cdot)$ are naturally equivalent* (*via* λ) *on the category of compact spaces. In particular, $K_M(\cdot)$ is independent of the factor M.*

PROOF. The functors K^* form a generalised cohomology theory, and this implies that $K^*(\cdot) \otimes_{\mathbf{Z}} \mathbf{R}$ do also. For clearly $K^n(\cdot) \otimes \mathbf{R}$ is a sequence of contravariant functors satisfying the excision axiom; the exactness axiom for $K(\cdot) \otimes \mathbf{R}$ follows since $\mathbf{R}$ is torsion-free and the continuity axiom follows since tensoring with $\mathbf{R}$ commutes with direct limits [**3**, pp. 33–34]. Let X be a compact space and let Per: $K(X) \to K(\mathbf{R}^2 \times X)$ and Per_M: $K_M(X) \to K_M(\mathbf{R}^2 \times X)$ denote the periodicity maps of K-theory and K_M-theory respectively. Then Per is given by taking the external product with the Bott element $b \in K(\mathbf{R}^2)$ [**4**, p. 118], and Per_M is the analogous external product for K_M-theory with the same element $b \in K(\mathbf{R}^2)$ [**6**, p. 426]. It follows from elementary properties of the external product (cf. [**6**, §4.11]) that the diagram

$$\begin{array}{ccc} K(X) \otimes \mathbf{R} & \xrightarrow{\mathrm{Per} \otimes \mathrm{id}} & K(\mathbf{R}^2 \times X) \otimes \mathbf{R} \\ \downarrow\lambda & & \downarrow\lambda \\ K_M(X) & \xrightarrow{\mathrm{Per}_M} & K_M(\mathbf{R}^2 \times X) \end{array}$$

commutes. Hence λ can be extended to give a natural transformation between the generalised cohomology theories $K^*(\cdot) \otimes \mathbf{R}$ and $K_M^*(\cdot)$. As observed above λ: $K^n(\mathrm{pt}) \otimes \mathbf{R} \to K_M^n(\mathrm{pt})$ is an isomorphism when $n = 0$; since every M-vector bundle on S^1 is trivial [**6**, Corollary 2, p. 404] it is also an isomorphism for $n = -1$, and it follows that λ is an isomorphism for all $n \in \mathbf{Z}$. The results now follow from the lemma in the introduction. □

It is a standard result in K-theory that for a compact space X, $K(X) \otimes \mathbf{R}$ is the direct sum of all the groups $H^p(X; \mathbf{R})$ for p even, where $H^p(X; \mathbf{R})$ denotes the pth Čech cohomology group of X with real coefficients. (This is a consequence of [**2**, p. 19] and the universal coefficient theorem. A more elementary proof is contained in [**1**, §3.2]; here, however, we have to invoke the Eilenberg-Steenrod uniqueness theorem to deduce that the H^p's are in fact Čech cohomology.) It now follows immediately from Proposition 2 that:

COROLLARY 3. *For any compact space X there is a natural isomorphism*

$$K_M(X) \cong \bigoplus \{ H^p(X; \mathbf{R}) : p \text{ even}, p \geq 0 \}.$$

2. Let X be a compact metric space. An extension of $C(X)$ relative to M is a unital $*$-monomorphism τ: $C(X) \to \mathfrak{A}(M)$. Two such extensions τ_1, τ_2 are equivalent if there is an inner automorphism α of M (which maps $\mathcal{K}(M)$ onto $\mathcal{K}(M)$ and so induces an automorphism $\bar{\alpha}$ of $\mathfrak{A}(M)$) with $\tau_2 = \bar{\alpha} \circ \tau_1$. The set $\mathrm{Ext}^M(X)$ of equivalence classes of extensions of $C(X)$ relative to M is a group (see [**12**]), is a homotopy invariant functor of the space X and can be used to define a generalised homology theory (see [**9**]). Cho also proves in [**9**] that Ext^M is naturally equivalent to $\mathrm{Hom}(\tilde{K}(S(\cdot)), \mathbf{R})$–and so is independent of M.

PROPOSITION 4. *For any compact metric space X there is a natural isomorphism*

$$\mathrm{Ext}^M(X) \cong \prod \{ H_p(X; \mathbf{R}) : p \text{ odd}, p \geq 1 \}$$

where $H_p(X, \mathbf{R})$ denotes the pth Čech homology group of X with real coefficients.

PROOF. First we suppose that X is a compact polyhedron. By the main theorem of [**9**], $\mathrm{Ext}^M(X) \cong \mathrm{Hom}(\tilde{K}(SX); \mathbf{R})$ where SX denotes the unreduced suspension of X. This in turn can be identified with $\mathrm{Hom}_{\mathbf{R}}(\tilde{K}(SX) \otimes_{\mathbf{Z}} \mathbf{R}, \mathbf{R})$, which is isomorphic to $\mathrm{Hom}_{\mathbf{R}}(\bigoplus_{p \text{ even}} \tilde{H}^p(SX; \mathbf{R}), \mathbf{R})$ by Corollary 3. Since X is a compact polyhedron $\mathrm{Hom}(\tilde{H}^p(SX); \mathbf{R}) \cong \tilde{H}_p(SX)$ [**13**, 23.14] and so $\mathrm{Ext}^M(X) \cong \prod_{p \text{ even}} \tilde{H}_p(SX; \mathbf{R})$. But $\tilde{H}_p(SX) \cong H_{p-1}(X)$, and we have the result for compact polyhedra. The general case now follows by observing that both Ext^M and $\prod H_p(., R)$ are continuous functors [**9**, Corollary 1].

REMARKS. Although the Čech homology theory $H_*(., G)$ with coefficients in an abelian group G satisfies the continuity axiom, it does not in general have a long exact sequence; the appropriate theory for compact metric spaces is Steenrod homology, denoted ${}^sH_*(., G)$. In addition to the seven Eilenberg-Steenrod axioms, ${}^sH_*$ satisfies the relative homeomorphism axiom and the cluster axiom (see [**15**] or [**16**]), and is characterised uniquely by these axioms [**16**, Theorem 3]. For an arbitrary coefficient group Čech homology satisfies all these axioms except exactness; however, when the coefficient group is $\mathbf{R}$, Čech homology is exact [**11**, Theorem IX.7.6] and so coincides with Steenrod homology on compact metric spaces. Thus the last proposition is valid with the Čech groups $H_*(X; \mathbf{R})$ replaced by the corresponding Steenrod groups

${}^{s}H_*(X;\,\mathbf{R})$. For further details on the relationships between Čech and Steenrod homology we refer to **[15]** and **[16]**; the Čech theory is discussed in detail in **[11]**.

In **[9]** Cho proves the Ext_*^M is a generalised Steenrod theory which is also continuous; hence Ext_*^M is also a generalised Čech theory. (Here by a generalised theory we mean one which satisfies all the appropriate axioms except dimension; the axioms for Čech homology are given in **[11**, Chapter X].) This is not the case for the Brown-Douglas-Fillmore theory Ext_*; it is a generalised Steenrod theory but is not continuous–in fact Brown **[7]** has shown that it fails to be continuous in the same way as the Steenrod homology theory. This is discussed in **[15]**.

ACKNOWLEDGEMENT. The author would like to thank Peter Fillmore for some helpful comments.

REFERENCES

1. M. F. Atiyah, *K-theory*, Benjamin, New York, 1967.
2. M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Differential Geometry, Proc. Sympos. Pure Math., vol. 3, Amer. Math. Soc., Providence, R. I., 1961, pp. 7–38.
3. M. F. Atiyah and I. G. MacDonald, *Introduction to commutative algebra*, Addison-Wesley, Reading, Mass., 1969.
4. M. F. Atiyah, *Algebraic topology and operators in Hilbert space*, Lecture Notes in Math., vol. 103, Springer-Verlag, Berlin and New York, 1969, pp. 101–121.
5. M. Breuer, *Fredholm theories in von Neumann algebras*. I, II, Math. Ann. **178** (1968), 243–254; **180** (1969), 314–325.
6. ______, *Theory of Fredholm operators and vector bundles relative to a von Neumann algebra*, Rocky Mountain J. Math. 3 (1963), 383–429.
7. L. G. Brown, *Operator algebras and algebraic K-theory*, Bull. Amer. Math. Soc. **81** (1975), 1119–1121.
8. L. G. Brown, R. G. Douglas and P. A. Fillmore, *Extensions of C^*-algebras and K-homology*, Ann. of Math. (2) **105** (1977), 265–324.
9. S. J. Cho, *Extensions relative to a II_∞-factor*, Proc. Amer. Math. Soc. (to appear).
10. J. Dixmier, *Les algèbres d'operateurs dans l'espace Hilbertien* (*Algèbres de von Neumann*), 2mè éd., Gauthier-Villars, Paris, 1969.
11. S. Eilenberg and N. E. Steenrod, *Foundations of algebraic topology*, Princeton Univ. Press, Princeton, N. J., 1952.
12. P. A. Fillmore, *Extensions relative to semi-finite factors*, Sympos. Math. **20** (1977), 487–496.
13. M. Greenberg, *Lectures on algebraic topology*, Benjamin, New York, 1967.
14. S. T. Hu, *Homotopy theory*, Academic Press, New York, 1959.
15. J. Kaminker and C. Schochet, *K-theory and Steenrod homology*: *Applications to the Brown-Douglas-Fillmore theory of operator algebras*, Trans. Amer. Math. Soc. **227** (1977), 63–107.
16. J. Milnor, *On the Steenrod homology theory*, mimeographed notes.
17. R. S. Palais, *Homotopy theory of infinite-dimensional manifolds*, Topology **5** (1966), 1–16.
18. I. M. Singer, *Some remarks on operator theory and index theory*, Lecture Notes in Math., vol. 575, Springer-Verlag, Berlin and New York, 1977, pp. 128–138.
19. E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.
20. J. L. Taylor, *Banach algebras and topology*, Algebras in Analysis (J. H. Williamson, Ed.,), Academic Press, London, 1975.
21. G. W. Whitehead, *Generalised homology theories*, Trans. Amer. Math. Soc. **102** (1962), 227–283.

SCHOOL OF MATHEMATICS, UNIVERSITY OF NEW SOUTH WALES, P. O. BOX 1, KENSINGTON, N.S.W. 2033, AUSTRALIA