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LOCAL p-SIDON SETS FOR LIE GROUPS

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ABSTRACT. It is shown that a compact Lie group admits no local p-Sidon sets of unbounded degree.

Let G be a compact group, and let $1 \leqslant p < 2$. A subset R of the dual of G is called a local p-Sidon set if there exists a constant B such that for every $\sigma \in R$ and for every $d_\sigma \times d_\sigma$ matrix A_σ,

$$\|A_\sigma\|_p \leqslant B d_\sigma^{1/p'} \|\operatorname{tr} A_\sigma \sigma(\cdot)\|_\infty. \tag{1}$$

THEOREM. *If G is a compact Lie group, and if R is a local p-Sidon set for G, then* $\sup\{d_\sigma | \sigma \in R\} < \infty$.

PROOF. We first note that, if G is an arbitrary compact group, R is a p-Sidon set for G, and if $r > 1$, then there exists a constant κ_r such that for all $\sigma \in R$

$$\|\chi_\sigma\|_r \leqslant \kappa_r d_\sigma^{2/p'} \tag{2}$$

where $\chi_\sigma(x) = \operatorname{tr}(\sigma(x))$.

To see this, we first use a simple duality argument to see that (1) is equivalent to: there exists a constant C such that for every $\sigma \in R$ and for every $d_\sigma \times d_\sigma$ matrix A_σ, there exists $g \in L^1(G)$ such that $\hat{g}(\sigma) = A_\sigma$, and $\|g\|_1 \leqslant C d_\sigma^{1/p'} \|A_\sigma\|_{p'}$. Thus for every $\sigma \in R$ and for every $d_\sigma \times d_\sigma$ unitary matrix W, there exists $g_W \in L^1(G)$ with $\hat{g}_W = W^*$, and $\|g_W\|_1 \leqslant C d_\sigma^{1/p'} \|W^*\|_{p'} = d_\sigma^{2/p'}$. Since $\chi_\sigma = g_W * (\operatorname{tr}(W \cdot \sigma(\cdot)))$ we have

$$\begin{aligned} \|\chi_\sigma\|_r &\leqslant \|g_W\|_1 \left(\int_G |\operatorname{tr}(W \cdot \sigma(x))|^r \, dx \right)^{1/r} \\ &\leqslant C d_\sigma^{2/p'} \left(\int_G |\operatorname{tr}(W \cdot \sigma(x))|^r \, dx \right)^{1/r}. \end{aligned}$$

Hence, integrating over the $d_\sigma \times d_\sigma$ unitary group, $\mathfrak{U}(d_\sigma)$ with respect to normalized Haar measure dW, and using Hölder's inequality, we obtain

$$\begin{aligned} \|\chi_\sigma\|_r &\leqslant C d_\sigma^{2/p'} \left(\int_G \int_{\mathfrak{U}(d_\sigma)} |\operatorname{tr}(W \cdot \sigma(x))|^r \, dW \, dx \right)^{1/r} \\ &= C d_\sigma^{2/p'} \left(\int_{\mathfrak{U}(d_\sigma)} |\operatorname{tr} W|^r \, dW \right)^{1/r}. \end{aligned}$$

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The last equality follows from the translation invariance of dW. It is easily established, however (cf. [2, (29.12)]), that there exists a bound K_r, independent of d_σ, for $(\int_{\mathfrak{U}(d_\sigma)}|\mathrm{tr}\, W|^r\, dW)^{1/r}$. Thus we obtain (2).

Suppose now that G is a compact Lie group. In [1, Theorem (5.4)], the following estimate is given for the r-norms of the irreducible characters; let $M_G \in \mathbf{R}$ be as in [1, (5.5)]. Then for $r > M_G$, there exists a constant κ_r such that

$$\kappa_r d_\sigma^{1-M_G/r} \leqslant \|\chi_\sigma\|_r. \tag{3}$$

From (2) and (3), it follows that, for all $r > M_G$, $\sup_{\sigma\in R} d_\sigma^{1-2/p'-M_G/r} < \infty$, and hence, since $p < 2$, $\sup_{\sigma \in R} d_\sigma < \infty$. $\square$

It follows that, if G is a compact semisimple Lie group, G has no infinite local p-Sidon sets.

It should be noted that a set R with $\sup\{d_\sigma | \sigma \in R\} < \infty$ is local Sidon and hence local p-Sidon for all p [3].

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