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ENDOHOMEOMORPHISMS DECOMPOSING A SPACE INTO DISJOINT COPIES OF A SUBSPACE

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ABSTRACT. The existence (conjectured by R. Levy in a private communication) of a space X and an endohomeomorphism, f, of βX, such that $f[X] = \beta X \setminus X$ is demonstrated. It is shown that if G is one of the topological groups $\mathbf{2}^\alpha$, $\mathbf{Q}^\alpha$, $\mathbf{R}^\alpha$ or $\mathbf{T}^\alpha$, where $\omega < \alpha$, then G has a dense C-embedded subgroup H and an autohomeomorphism, f, such that G is the union of disjoint sets, A_0 and A_1, where for $\{i, j\} = \{0, 1\}$ $f[A_i] = A_j$, and A_i is a union of cosets of H.

The existence (conjectured by R. Levy in a private communication) of a space X and an endohomeomorphism, f, of βX, such that $f[X] = \beta X \setminus X$ is demonstrated. It is shown that if G is one of the topological groups $\mathbf{2}^\alpha$, $\mathbf{Q}^\alpha$, $\mathbf{R}^\alpha$ or $\mathbf{T}^\alpha$, where $\omega < \alpha$, then G has a dense C-embedded subgroup H and an autohomeomorphism, f, such that G is the union of disjoint sets, A_0 and A_1, where for $\{i, j\} = \{0, 1\}$ $f[A_i] = A_j$, and A_i is a union of cosets of H. These results, which continue (and duplicate in part) a remark of Glicksberg **[G59]**, have appeared in **[O'C76]**.

NOTATION. We denote the nonnegative integers, the first infinite ordinal and the corresponding countable discrete space by ω. The topological groups $\mathbf{2}$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{T}$ are $\{0, 1\}$, the integers, the rationals, the reals and the circle group $(\mathbf{R}/\mathbf{Z})$, respectively. We denote the domain and the range of a function f by $\operatorname{Dom} f$ and $\operatorname{Rng} f$, respectively. Let f be a function, and let A be a subset of $\operatorname{Dom} f$. An f-decomposition of A is a pair $\langle A_0, A_1\rangle$ of complementary subsets of A, such that $A_i \cap f[A_i] = \varnothing$ for $i = 0, 1$. If in addition the set A satisfies $f[A] = A$, then an f-bisection of A is an f-decomposition $\langle A_0, A_1\rangle$ with $f[A_0] = A_1$ and $f[A_1] = A_0$. Otherwise the notation is that of **[GJ60]** and **[CN74]**.

Our first result is an application of the Katětov Lemma on Three Sets of which we give only a skeletal proof. A detailed proof can be found in **[B64]**, **[K67]**, **[CN74]** and **[W74]**.

LEMMA 1. *Let f be a function with* $\operatorname{Dom} f \supseteq \operatorname{Rng} f$, *and let A be a subset of* $\operatorname{Dom} f$.

(a) *If f^{2k+1} has no fixed points for $k = 0, 1, \ldots$, then A has an f-decomposition.*

(b) *If f has no fixed points, then $A = A_0 \cup A_1 \cup A_2$, where $\{A_0, A_1, A_2\}$ is a pairwise disjoint family, and $A_i \cap f[A_i] = \varnothing$ for $i = 0, 1, 2$.*

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PROOF. Define $\backsim$ by $x \backsim y$, whenever $f^n(x) = f^m(y)$ for some nonnegative integers m and n. Clearly $\backsim$ is an equivalence relation. To produce a decomposition of the claimed type for A it suffices to produce an appropriate decomposition of the equivalence class, $x/\backsim$ where x ranges over a complete set of representatives of A. The reader may verify, that in case (a)

$$[x]_0 = \{y \in x/\backsim: m \equiv n \pmod 2, \text{ whenever } f^n(x) = f^m(y)\}$$

$$[x]_1 = \{y \in x/\backsim: m \not\equiv n \pmod 2, \text{ whenever } f^n(x) = f^m(y)\}$$

yields an f-decomposition of $x/\backsim$.

A slight modification of this argument proves (b): Should $x/\backsim$ not contain a fixed point of f^{2k+1} for any nonnegative integer k, then $[x]_0$ and $[x]_1$ are defined as above, and $[x]_2 = \varnothing$; should $x/\backsim$ contain a fixed point of f^{2k+1}, where k is minimal for this x, then we choose x, the class representative, to be one of the $(2k + 1)$ fixed points of f^{2k+1} in this equivalence class, and we define

$[x]_0 = \{y - x/\backsim: n \equiv 0 \pmod 2$, whenever n is minimal with $f^n(y) = x\} \setminus \{x\}$.

$[x]_1 = \{y - x/\backsim: n \equiv 1 \pmod 2$, whenever n is minimal with $f^n(y) = x\}$,

$[x]_2 = \{x\}$.

The verification of the details is left to the reader.

COROLLARY. *Under the hypotheses of case* (a):

(i) *If* $n \equiv m \pmod 2$, *whenever* x *and* $y \in \operatorname{Dom} f \setminus \operatorname{Rng} f$, *and* $f^n(x) = f^m(y)$, *then* Dom f *has an* f*-decomposition of the form* $\langle X, f[X]\rangle$.

(ii) *If* Dom f = Rng f, *then* Dom f *has an* f*-bisection*.

(iii) *Let* $x \in \operatorname{Dom} f$, *and let* $\langle A_0, A_1\rangle$ *be an* f*-decomposition of* Dom f, *then* $\{x/\backsim \cap A_0, x/\backsim \cap A_1\} = \{[x]_0, [x]_1\}$.

PROOF. For (i): choose, whenever possible, a member of $\operatorname{Dom} f \setminus \operatorname{Rng} f$ as class representative. Then $[x]_0 \supseteq x/\backsim \cap (\operatorname{Dom} f \setminus \operatorname{Rng} f)$, so $[x]_1 = f[[x]_0]$.

Parts (ii) and (iii) are obvious.

DEFINITION. A subset D of a topological space is said to be strongly discrete if for each d in D there is a neighborhood, U_d, of d such that $\{U_d: d \in D\}$ is a pairwise disjoint family.

We note that if D is a strongly discrete subset of $\beta\alpha$, then D is C^*-embedded in $\beta\alpha$ **[CN74]**, and any one-to-one function, $g: \gamma \to D$, has an extension $f: \beta\gamma \to \beta\alpha$ which is a homeomorphism.

THEOREM 1. *Let* $g: \omega \to \beta\omega \setminus \omega$ *be a one-to-one function with strongly discrete image, and let* $f: \beta\omega \to \beta\omega \setminus \omega$ *be its continuous extension. The space* $\beta\omega$ *has an* f*-decomposition* $\langle X, \beta X \setminus X\rangle$.

PROOF. By a theorem of Frolík **[F67]** no power of f has a fixed point (if we do not wish to use this result of Frolík, then all we need do is to add the condition that $g(2n) \in \operatorname{cl}\{2k + 1: k < \omega\}$ and $g(2n + 1) \in \operatorname{cl}\{2k: k < \omega\}$

for all n in ω: this prevents each odd power of f from having a fixed point). There is by the corollary to the Katětov Lemma a set X such that $X \supseteq \operatorname{Dom} f \setminus \operatorname{Rng} f \supseteq \omega$, and $\langle X, f[X] \rangle$ is an f-decomposition of $\beta\omega = \beta X$.

This result has an obvious generalization to the Stone-Čech compactification of an arbitrary infinite cardinal. However, it is not the case that to construct an example all one needs is a homeomorphism $g\colon X \to \beta X \setminus X$ no (odd) power of which has a fixed point, which, furthermore, has a continuous extension $f\colon \beta X \to \beta X \setminus X$, where f is a homeomorphism. We have the following counterexample.

Let W be the set of countable ordinals with the order topology. It is well known **[GJ60]** that $\beta W = W^*$–the one point compactification of W. Let $W^* = W \cup \{ * \}$. Let $Y = W \times \Pi\{ W^*\colon 0 < n < \omega\}$. Clearly $\beta Y = \Pi\{ W^*\colon n < \omega\}$. Define $g\colon Y \to \beta Y \setminus Y$ as follows:

$$g(\langle w_0, w_1, \ldots \rangle) = \langle *, w_0, w_1, \ldots \rangle.$$

Then g has a continuous extension f which is defined by the same rule but with domain βY, and f is a homeomorphism. But $\langle *, *, \ldots \rangle$ is a fixed point of f, so βY does not have an f-decomposition.

It may be worth mentioning that Y has the property that βY is homeomorphic to $\beta Y \setminus Y$. Also, if we had uncountably many factors in the definition of Y, and if g were defined as above, then we would have a homeomorphism f extending g. This time f would have a set of fixed points, which would be homeomorphic to βY.

Definition. Let $X = \Pi\{X_i\colon i \in I\}$, let $p \in X$, and let $\omega < \gamma$. Then $\Sigma_\gamma(p, X)$ is defined as follows:

$$\Sigma_\gamma(p, X) = \{ y \in X\colon |\{ i \in I\colon p_i \neq y_i \}| < \gamma \}.$$

We will write $\Sigma_\gamma(p)$ when the space X is obvious from the context.

If each of the factors of X is a separable metric space, and if α is uncountable, then $\Sigma_\alpha(p)$ is dense and C-embedded **[Cn59]**.

Observation. Let F be a topological field, and let α be uncountable. The vector space F^α is then equipped with the product topology. Translation by $\langle 1, 1, \ldots \rangle$ is denoted by f. We observe by the Katětov Lemma and its corollary that the space F^α has an f-bisection if and only if F has even characteristic (including characteristic zero). We can sharpen this result.

Theorem 2. (Glicksberg). *Let α be uncountable, and let f be translation by $\langle 1, 1, \ldots \rangle$ on the space $\mathbf{2}^\alpha$ regarded as a vector space over the field $\mathbf{2}$. Then $\mathbf{2}^\alpha$ has an f-bisection $\langle A_0, A_1 \rangle$, where A_0 is a dense C-embedded, maximal vector subspace, M, of $\mathbf{2}^\alpha$. So βM is $\mathbf{2}^\alpha$, and the f-bisection has the form $\langle M, \beta M \setminus M \rangle$.*

Proof. Clearly $\Sigma_\alpha(\langle 0, 0, \ldots \rangle)$ is a dense vector subspace of $\mathbf{2}^\alpha$, which misses $\langle 1, 1, \ldots \rangle$. By Zorn's Lemma there is a subspace M of $\mathbf{2}^\alpha$, which is maximal with respect to both missing $\langle 1, 1, \ldots \rangle$ and containing

$\Sigma_\alpha(\langle 0, 0, \ldots \rangle)$. It can be shown that M has the required properties.

Clearly in Theorem 2 M is just a maximal subgroup of $\mathbf{2}^\alpha$. The language of vector spaces is used to emphasize the relation with Theorem 3. The basic idea of both these proofs is to find some subgroup M of the product F^α such that F^α / M has an appropriate decomposition.

THEOREM 3. *Let F be a subfield of* $\mathbf{R}$. *Let α be uncountable, and let f denote translation by $\langle 1, 1, \ldots \rangle$ on the vector space F^α. Then F^α has an f-bisection $\langle A_0, A_1 \rangle$, where A_0 and A_1 are unions of cosets of a dense, C-embedded, maximal subspace of F^α.*

PROOF. Arguing as in Theorem 2 we can find a dense C-embedded, maximal subspace M of F^α, which misses $\langle 1, 1, \ldots \rangle$. Define R, S, A_0 and A_1 as follows:

$R = \bigcup \{[2n, 2n + 1) \cap F: n \in \mathbf{Z}\}$,

$S = \bigcup \{[2n + 1, 2n + 2) \cap F: n \in \mathbf{Z}\}$,

$A_0 = M + R^\alpha$, and

$A_1 = M + S^\alpha$.

Clearly R and S are complementary subsets of F, and A_0 and A_1 are as required.

The group $\mathbf{T}^\alpha$ has a decomposition similar to the decomposition of Theorem 3. There does not seem to be a proof along the lines of the proofs of Theorems 2 and 3. A more elaborate version of the decomposition of Theorem 3 seems necessary.

THEOREM 4. *Let E be a dense subgroup of* $\mathbf{R}$, *which contains* 1. *Let α be uncountable, and let f be translation by $\langle 1, 1, \ldots \rangle$ on the topological group E^α. Then E^α has an f-bisection $\langle A_0, A_1 \rangle$, where A_0 and A_1 are dense and C-embedded in E^α.*

PROOF. Let M be a maximal subgroup of $\mathbf{2}^\alpha$ as constructed in Theorem 2. We define (using the addition of E^α) the sets X, Y, V, W, A_0 and A_1 as follows:

$X = \Sigma_\alpha(\langle 0, 0, \ldots \rangle, ([0, 1) \cap E)^\alpha)$,

$Y = ([0, 1] \cap E)^\alpha \setminus X$,

$V = X + M + (2\mathbf{Z})^\alpha$,

$W = Y + M + (2\mathbf{Z})^\alpha$,

$A_0 = V \cup (W + \langle 1, 1, \ldots \rangle)$,

$A_1 = W \cup (V + \langle 1, 1, \ldots \rangle)$.

If $x \in E^\alpha$, then there is a unique p in $\mathbf{2}^\alpha$ and a unique q in $(2\mathbf{Z})^\alpha$ such that $x - p - q \in ([0, 1) \cap E)^\alpha$. Clearly

$$\Sigma_\alpha(\langle 0, 0, \ldots \rangle, E^\alpha) \subseteq X + \Sigma_\alpha(\langle 0, 0, \ldots \rangle, \mathbf{2}^\alpha) + \Sigma_\alpha(\langle 0, 0, \ldots \rangle, (2\mathbf{Z})^\alpha) \subseteq V.$$

Hence V is dense and C-embedded in E. Furthermore the family of sets

$$\{V, V + \langle 1, 1, \ldots \rangle, W, W + \langle 1, 1, \ldots \rangle\}$$

is pairwise disjoint, since the family of sets

$$\{X + M, X + M + \langle 1, 1, \ldots \rangle, Y + M, Y + M + \langle 1, 1, \ldots \rangle\}$$

is pairwise disjoint. Since $A_i + \langle 1, 1, \ldots \rangle = A_j$ for $\{i, j\} = \{0, 1\}$, we see that $\langle A_0, A_1 \rangle$ is as required.

THEOREM 5. *Let G be a dense subgroup of* $\mathbf{T}$, *which contains* $-1 = \exp(\pi i)$. *Let α be uncountable, and let f be translation (multiplication) by $\langle -1, -1, \ldots \rangle$ on G^α. Then G^α has an f-bisection $\langle B_0, B_1 \rangle$, where B_0 and B_1 are dense and C-embedded in G^α.*

PROOF. The αth power, H, of the canonical homomorphism, h: $R \to \mathbf{T}$ is a continuous homomorphism from $\mathbf{R}^\alpha$ onto $\mathbf{T}^\alpha$. Let E be the inverse image of G under h. Then E is a dense subgroup of $\mathbf{R}$, which contains 1, such that $h[E] = G$. Let A_0 and A_1 be as constructed in Theorem 4. Clearly

$$H[A_0] \cap H[A_1] = \varnothing.$$

Since $A_0 \supseteq \Sigma_\alpha(\langle 0, 0, \ldots \rangle, E^\alpha)$,

$$H[\Sigma_\alpha(\langle 1, 1, \ldots \rangle, E^\alpha)] = \Sigma_\alpha(\langle -1, -1, \ldots \rangle, G^\alpha),$$

and

$$H[\Sigma_\alpha(\langle 0, 0, \ldots \rangle, E^\alpha] = \Sigma_\alpha(\langle 1, 1, \ldots \rangle, G^\alpha),$$

then $\langle B_0, B_1 \rangle$ defined by $B_0 = H[A_0]$, and $B_1 = H[A_1]$ is as required.

[CO'C79] will contain results of related interest. In particular sharper characterizations are obtained for first countable, realcompact spaces X of (i) continuous functions f: $X \to \beta X$ and (ii) spaces Y with $X \subset Y \subset \beta X$, and with $g[Y] = \beta Y \setminus Y$, where g is an endohomeomorphism of βX.

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