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CHARACTERIZING SERIES FOR FAITHFUL D.G. NEAR RINGS

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ABSTRACT. Let (R, S) be a distributively generated near ring satisfying $(R, S) \subseteq (E(G), \mathrm{End}(G))$ and $S \subseteq \mathrm{End}(G)$ for some group G, endomorphism near ring $E(G)$, and subsemigroup S of the endomorphisms of G, $\mathrm{End}(G)$. The radicals $J(R)$ of (R, S) are characterized in terms of series of subgroups of G. We assume S contains the inner automorphisms of G and obtain two main results on characterizing series. (1) If G satisfies both chain conditions on S subgroups then a unique minimal characterizing series exists. (2) If G is finite, then both maximal and minimal characterizing series exist, are unique, and are themselves characterized in G.

0. Introduction. Let $(G, +)$ denote a group, $E(G)$ the near ring generated by the endomorphisms of G, End G. We consider distributively generated (d.g.) near rings $(R, S) \subseteq (E(G), \text{End } G)$, where $S \subseteq \text{End } G$. We seek to characterize radicals $J(R)$ of R in terms of series of subgroups of G. If $S \supseteq \text{Inn } G$, the inner automorphisms of G, then a unique minimal characterizing series exists whenever G satisfies both chain conditions on S-subgroups. If G is finite, then unique minimal and maximal characterizing series exist and can be given "internal" characterizations.

Many basic definitions and results can be found in G. Pilz **[11]**, which we will follow for the most part, except that we use left near rings instead of right near rings. If the near ring R is distributively generated by the semigroup S, then we denote it by (R, S). We say that (R, S) has a faithful d.g. representation θ on the group G if there exists a monomorphism θ: $(R, S) \to (E(G), \text{End } G)$ such that $S\theta \subseteq \text{End } G$. A d.g. near ring is called faithful if it has a faithful d.g. representation θ on some group G. There are three d.g. near rings associated with a group which are of special interest. Define Inn G, Aut G, and End G to be, respectively, the semigroups of all inner automorphisms, all automorphisms, and all endomorphisms of the group G. These semigroups generate the d.g. near rings $(I(G), \text{Inn } G)$, $(A(G), \text{Aut } G)$ and $(E(G), \text{End } G)$.

1. Characterizing series for a radical. Let (R, S) be a faithful d.g. near ring with a faithful representation on a group G.

DEFINITION 1.1. An (R, S) *series of* G is a series

$$\mathrm{C}\colon G = G_0 \supset G_1 \supset \cdots \supset G_n = \{0\}$$

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of G such that G_{i+1} is an ideal of G_i for $0 \leqslant i \leqslant n-1$, i.e. an R-invariant normal subgroup. We say **C** has length n.

An important concept in this context is the annihilator of an (R, S) series of G.

DEFINITION 1.2. Let **C** be an (R, S) series of G. Then the *annihilator of* **C** is defined by

$$\mathrm{Ann}(\mathbf{C}) = \{\alpha \in R;\ G_i\alpha \subseteq G_{i+1}, 0 \leqslant i \leqslant n-1\}.$$

The following result is easily proved.

LEMMA 1.3. *Let* **C** *be an* (R, S) *series of* G. *Then* Ann(**C**) *is an ideal of* R *and* $(\mathrm{Ann}(\mathbf{C}))^n = \{0\}$.

PROOF. $\mathrm{Ann}(\mathbf{C}) = \bigcap_{i=0}^{n-1} \mathrm{Ann}(G_i/G_{i+1})$ proves the first statement. The second follows if we observe that $(\mathrm{Ann}(\mathbf{C}))^j$ maps G_i to G_{i+j}, taking $G_{i+j} = \{0\}$ if $i + j \geqslant n$.

We use the convention that if A and B are two sets then $AB = \{\alpha\beta;\ \alpha \in A, \beta \in B\}$, where B is a subset of R, A a subset of R or of G.

Let $J = J(R)$ be a radical of R. This could be any of the radicals that have been defined for near rings. Betsch [2] and Laxton [4] were among the first to study near ring radicals, and there is a good account of several radicals in Johnson [3] and Pilz [11].

DEFINITION 1.4. Let $J = J(R)$ be a radical of R. Then **C** is called a *characterizing series for* J if and only if (i) $\mathrm{Ann}(\mathbf{C}) = J$, (ii) $G_iJ \not\subseteq G_{i+2}$ for $0 \leqslant i \leqslant n-2$.

Note that since by Lemma 1.3, $(\mathrm{Ann}(\mathbf{C}))^n = \{0\}$, a radical has a characterizing series only if it is nilpotent.

Denote by Id (X) the ideal generated by the subset X of G. Then there are two obvious candidates for characterizing series:

$$\mathbf{L}\colon G = L_0, \qquad L_{i+1} = \mathrm{Id}(L_iJ) \quad \text{for } i \geqslant 0, \tag{1.5}$$

$$\mathbf{K}\colon G = K_0, \qquad K_i = \mathrm{Id}(GJ^i) \quad \text{for } i \geqslant 1. \tag{1.6}$$

While it is obvious that $L_i \supseteq L_{i+1}$ and $K_i \supseteq K_{i+1}$ for all $i \geqslant 0$, there is no guarantee that either will reach the identity or that either sequence of ideals forms a series.

LEMMA 1.7. **K** *is a series of length* n *if and only if* J *is nilpotent of nilpotency class* n.

The proof is obvious, once we remember that (R, S) has a faithful representation θ on G.

LEMMA 1.8. $L_i \supseteq K_i$ *for* $i \geqslant 0$.

PROOF. The result is true for $i = 0$. Assume that it is true for i. Then $GJ^i \subseteq K_i \subseteq L_i$. Hence

$$GJ^{i+1} \subseteq K_iJ \subseteq L_iJ \subseteq L_{i+1}.$$

So $\mathrm{Id}(GJ^{i+1}) = K_{i+1} \subseteq L_{i+1}$ as L_{i+1} is an ideal. This proves the result by induction.

We now relate **L** to characterizing series.

THEOREM 1.9. *Let* **C** *be an* (R, S) *series of* G *such that* $J \subseteq \mathrm{Ann}(\mathbf{C})$. *Then* $G_i \supseteq L_i$ *for* $i \geqslant 0$.

PROOF. Certainly $G_0 \supseteq L_0$. Assume that $G_i \supseteq L_i$. Then $G_iJ \subseteq G_{i+1}$ since $J \subseteq \mathrm{Ann}(\mathbf{C})$. So

$$G_{i+1} \supseteq G_iJ \supseteq L_iJ.$$

Thus $\mathrm{Id}(L_iJ) \subseteq G_{i+1}$ as G_{i+1} is an ideal. Hence $L_{i+1} \subseteq G_{i+1}$ and the result follows by induction.

THEOREM 1.10. *Let* J *be nilpotent of nilpotency class* n. *If* **C** *is an* (R, S) *series of length* n *such that* $\mathrm{Ann}(\mathbf{C}) = J$, *then* **C** *is a characterizing series for* J.

PROOF. We only have to check that $G_iJ \not\subseteq G_{i+2}$ for $0 \leqslant i \leqslant n-1$. But if for some i, we have $G_iJ \subseteq G_{i+2}$, then $GJ^{n-1} \subseteq G_n = \{0\}$. Hence $J^{n-1} = \{0\}$ as R has a faithful representation θ on G and this contradicts the hypothesis that J is nilpotent of nilpotency class n.

COROLLARY 1.11. *If* **C** *is a characterizing series for* J, *then* $G_i \supseteq L_i$ *for all* $i \geqslant 0$.

PROOF. This follows immediately from Theorem 1.9.

Theorem 1.10 shows that all characterizing series for J have the same length, namely the nilpotency class of J.

DEFINITION 1.12. Let $J = J(R)$ be a radical for which characterizing series exist. Then an (R, S) series **D** of G,

$$\mathbf{D}\colon G = H_0 \supset H_1 \supset H_2 \supset \cdots \supset H_{n-1} \supset H_n = \{0\}$$

is called a *minimal characterizing series for* J if whenever

$$\mathbf{C}\colon G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_{n-1} \supset G_n = \{0\}$$

is a characterizing series for J, then $G_i \supseteq H_i$ for $0 \leqslant i \leqslant n$.

We now show that if J contains all nilpotent ideals of R, and has a characterizing series, then the chain **L** defined by (1.5) is the unique minimal characterizing series for J. In this case **L** is meant to indicate lower. The relationship of **L** to characterizing series for J is similar to that of the lower central series of a group to central series of that group.

Whatever the radical, **L** is defined as a sequence of ideals of G, although it will not be a series unless the radical is nilpotent. Some light is thrown on the situation by the following result.

LEMMA 1.13. *If* $L_iJ \subseteq L_{i+2}$ *for some* i, *then* $L_{i+1} = L_{i+k}$ *for all* $k \geqslant 1$.

PROOF. $L_iJ \subseteq L_{i+2}$ forces $\mathrm{Id}(L_iJ) = L_{i+1} \subseteq L_{i+2}$ since L_{i+2} is an ideal. Hence

$$L_{i+1} = L_{i+2} = \mathrm{Id}(L_{i+1}J).$$

If $L_{i+k} = L_{i+1}$ then $L_{i+k+1} = \mathrm{Id}(L_{i+k}J) = \mathrm{Id}(L_{i+1}J) = L_{i+1}$. So, by induction, $L_{i+k} = L_{i+1}$ for all $k \geqslant 1$.

COROLLARY 1.14. *Let* **L** *be an* (R, S) *series of* G. **L** *is a characterizing series for* J *if and only if* $\mathrm{Ann}(\mathbf{L}) = J$.

PROOF. This follows directly from the previous result.

As we know that $J \subseteq \mathrm{Ann}(\mathbf{L})$ by definition, we need only check that $\mathrm{Ann}(\mathbf{L}) \subseteq J$ to be able to use Corollary 1.14.

THEOREM 1.15. *If* $J(R)$ *contains all the nilpotent ideals of* R, *and if* **L** *is a series of length* n, *then* J *is nilpotent of nilpotency class* n *and* **L** *is the unique minimal characterizing series for* J.

PROOF. If **L** has length n, then $(\mathrm{Ann}\,(\mathbf{L}))^n = \{0\}$. So $\mathrm{Ann}\,(\mathbf{L}) \subseteq J$ by hypothesis. But, $J \subseteq \mathrm{Ann}\,(\mathbf{L})$ so $J = \mathrm{Ann}\,(\mathbf{L})$ and J is nilpotent of class n. By Corollary 1.14, **L** is a characterizing series and from Corollary 1.11 it follows that **L** is the unique minimal characterizing series for J.

Lemma 1.8 shows that **K** is a sequence of ideals which lies below **L**, i.e. $K_i \subseteq L_i$ for all $i \geqslant 0$. Also if J is nilpotent of nilpotency class n, then **K** is a series of length n. An unsolved problem is whether $J \subseteq \mathrm{Ann}\,(\mathbf{K})$.

In [**3**] M. Johnson considers five different radicals for $E(G)$ for G finite, four of them being the J_2, J_0, N and P radicals of G. Pilz [**11**], the other being the primitive radical (see Beidleman [**1**] and Betsch [**2**]). The above work enables us to generalize Theorem 16 of Johnson [**3**].

THEOREM 1.16. *Let* (R, S) *have a faithful d.g. representation* θ *on* G *so that* $S\theta \supseteq \mathrm{Inn}\, G$. *If* G *possesses a finite* S *composition series, then the* J_2, J_0, N *and* P *radicals coincide. If, in addition,* $R/J_2(R)$ *satisfies the descending chain condition on* R *subgroups, then the primitive radical coincides with the other four.*

PROOF. Note that G possesses a finite S composition series if and only if G satisfies the maximum and minimum condition on S subgroups, and that under the hypotheses of the theorem all S subgroups are normal, hence all R subgroups are ideals. By 5.66, Pilz [**11**], all four radicals contain all nilpotent ideals. We now show that **L** is a series. Consider L_i. Pick a maximal proper ideal of L_i, say M. Then an R subgroup H/M of L_i/M arises from an R subgroup H of G. But then H is an ideal of G and so H/M is an ideal of L_i/M. Thus L_i/M has no nontrivial R subgroups, and so $(L_i/M)J = \{0\}$ for any of the four radicals. Hence $L_iJ \subseteq M$ and so $L_{i+1} \subseteq M$. Thus the sequence of ideals defined in (1.5) is strictly descending. The minimum condition on S subgroups implies that $L_n = \{0\}$ for some n. This enables us to apply Theorem 1.15 to get the first part of our result. Finally, apply Theorem 4 of Beidleman [**1**] for the final part.

2. Characterizing series in finite groups. To progress further, either with an internal characterization of the unique minimal characterizing series, or the existence of a unique maximal characterizing series, we need to restrict ourselves to finite groups. So from now on we assume that G is a finite group, and that (R, S) has a faithful d.g. representation θ on G such that $S\theta \supseteq \operatorname{Inn} G$. The key result is the following extension of Proposition 5 of Johnson [3].

THEOREM 2.1. *Let G be a finite group, (R, S) have a faithful d.g. representation θ on G such that $S\theta \supseteq \operatorname{Inn} G$. Then $J(R) = \{0\}$ if and only if G is the direct sum of minimal S subgroups.*

Here $J(R)$ is any one of the five radicals mentioned in the previous section. The proof of this result is a straightforward extension of Johnson's proof. We apply it in the following situation.

THEOREM 2.2. *We assume the hypothesis of Theorem 2.1. Let H be an S subgroup of G. Then the ideal of H generated by HJ is the intersection of all the maximal S subgroups of H, where J is any one of the five radicals mentioned earlier.*

We will use the notation J for any one of the five radicals of R mentioned earlier for the rest of this paper.

PROOF. Let K be the intersection of all the maximal S subgroups of H, say $K = \bigcap_{i=1}^{n} M_i$, M_i a maximal S subgroup of H. Since H is an S subgroup, then $S\theta \subseteq \operatorname{End} G$ and $HS \subseteq H$ implies that the elements of $S\theta$ restricted to H are endomorphisms of H. Hence there is a d.g. representation of (R, S) on H induced by θ. In that representation the image of S contains $\operatorname{Inn} H$, since $S\theta \supseteq \operatorname{Inn} G$ and $\operatorname{Inn} G$ restricted to H contains $\operatorname{Inn} H$. Since H/M_i is a minimal (R, S) subgroup, we deduce that $HJ \subseteq M_i$ for $1 \leqslant i \leqslant n$ and so that $HJ \subseteq \bigcap_{i=1}^{n} M_i = K$. Let L be the ideal of H generated by HJ. Then H/L is annihilated by J. Since S maps H/L into itself, there is a homomorphism ϕ taking $(R, S) \to (E(H/L), \operatorname{End} H/L)$ with $S\phi \subseteq \operatorname{End} H/L$ which gives a d.g. representation of (R, S) on H/L. As above $S\phi \supseteq \operatorname{Inn} H/L$. Also $\operatorname{Ker} \phi \supseteq J$ by definition of L. Let $T = \operatorname{Ker} \phi$. Then $(R, S)/T$ is a homomorphic image of $(R, S)/J$ and by [**11**, 5.31], $(R, S)/T$ has a trivial radical. We can now apply Theorem 2.1 to deduce that H/L is the direct sum of minimal $S\phi$ groups, hence the direct sum of minimal S groups. But then L is the intersection of the maximal S subgroups of H which contain L. Hence $K = L$, the result we want.

COROLLARY 2.3. *Under the hypotheses of Theorem 2.1, let $\mathbf{L}$ be the minimal characterizing series for $J(R)$. Then L_{i+1} is the intersection of all the maximal S subgroups of L_i for $0 \leqslant i \leqslant n-1$.*

Identify (R, S) with its image in $E(G)$. If $S = \operatorname{Inn} G$, then S subgroups are just normal subgroups of G. If $S = \text{Aut}\, G$, then S subgroups are the characteristic subgroups of G. If $S = \text{End}\, G$, then S subgroups are the fully invariant subgroups of G.

DEFINITION 2.4. Let **H** be a characterizing series for J, where H is

$$G = H_0 \supset H_1 \supset \cdots \supset H_n = \{0\}.$$

Then we say **H** is the *maximal characterizing series for* J if whenever **C** is a characterizing series for J, then $H_i \supseteq G_i$ for all $i \geqslant 0$.

Note that by Theorem 1.10, all characterizing series have the same length. Note also that the maximal characterizing series is unique if it exists. It corresponds to the upper central series of a nilpotent group (**H** for higher) in the same way that **L** corresponds to the lower central series.

Let G be a finite group, (R, S) have a faithful d.g. representation θ on G such that $S\theta \supseteq \text{Inn}\, G$. Define **U** by **U**: $G = U_0 \supset U_1 \supset \cdots \supset U_m = \{0\}$, where $U_m = \{0\}$ and if U_j is defined for $j > i$, then U_i/U_{i+1} is the sum of all minimal S subgroups of G/U_{i+1}. As G/U_{i+1} is finite there always exist nontrivial minimal S subgroups of G/U_{i+1}. Since G is finite, **U** is an (R, S) series of G.

LEMMA 2.5. $J(R) = \text{Ann}(\mathbf{U})$.

PROOF. Certainly $U_iJ \subseteq U_{i+1}$ for $0 \leqslant i \leqslant m-1$, since (R, S) acts on U_i/U_{i+1} and J annihilates all minimal S subgroups. Hence $J \subseteq \text{Ann}(\mathbf{U})$. But $\text{Ann}(\mathbf{U})$ is nilpotent, so $\text{Ann}(\mathbf{U}) \subseteq J$. Thus $J = \text{Ann}(\mathbf{U})$.

THEOREM 2.6. **U** *is* **H** *the maximal characterizing series for* J.

PROOF. Let **C** be a characterizing series for J. We prove by induction on i that $U_{m-i} \supseteq G_{n-i}$. Certainly $U_m \supseteq G_n$. Assume that the result is true for i, and that $U_{m-i} \supseteq G_{n-i}$. For ease of notation write

$$X = U_{m-i-1}, \quad Y = U_{m-i}, \quad U = G_{n-i-1}, \quad V = G_{n-i}.$$

Since $Y \supseteq V$ and $UJ \subseteq V$, we deduce that

$$(Y + U)J \subseteq Y \quad \text{as} \quad (Y + U)/Y \cong U/(Y \cap U)$$

is a homomorphic image of U/V. We now apply Theorem 2.1 to the (R, S) module $(Y + U)/Y$. Since $J(R)$ annihilates $(Y + U)/Y$, we must have that $(Y + U)/Y$ is the direct sum of minimal S subgroups. Hence $Y + U \subseteq X$ by definition of the **U** series. Thus $U \subseteq X$ and we have proved by induction that $U_{m-i} \supseteq G_{n-i}$. In particular $U_{m-n} \supseteq G_0 = G$. So $m \leqslant n$. But $\text{Ann}(\mathbf{U}) = J$ has nilpotency class n. So $m \geqslant n$. An application of Theorem 1.10 now suffices to prove the result that we want.

If G is a finite group, with minimal characterizing series **L** and maximal characterizing series **H**, then $H_i \supseteq L_i$ for all $i \geqslant 0$. In general these two ideals will be distinct. In cases where $I(G)$, $A(G)$ and $E(G)$ have been calculated, these results enable us to find J more easily and to calculate its nilpotency class exactly. See Lyons **[5]**, Malone and Lyons **[7]**, **[8]** and **[9]**, Malone **[6]**, and Meldrum **[10]**.

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