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A PSEUDO DIFFERENTIAL OPERATOR WHICH SHIFTS THE WAVE FRONT SET

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ABSTRACT. The note concerns pseudo differential operators in the classes $L^m_{\rho,1}$, $0 < \rho < 1$. These operators are pseudo-local, but they can displace the wave front set of distributions, as we show by means of an example in $L^0_{1,1}$.

1. Classes $L^m_{\rho,\delta}$ and pseudo-local property. Let Ω be an open set in $\mathbf{R}^n$. Denote by $S^m_{\rho,\delta}(\Omega \times \Omega \times \mathbf{R}^n)$, $0 \leqslant \rho \leqslant 1$, $0 \leqslant \delta \leqslant 1$, the class of all $a(x, y, \xi) \in C^\infty(\Omega \times \Omega \times \mathbf{R}^n)$ such that for every compact set $K \subset \Omega \times \Omega$ and all multiorders α, β, γ the estimate

$$\left|D_x^\alpha D_y^\beta D_\xi^\gamma a(x, y, \xi)\right| \leqslant c_{\alpha,\beta,\gamma,K}(1 + |\xi|)^{m - \rho|\gamma| + \delta|\alpha+\beta|} \tag{1}$$

is valid in $K \times \mathbf{R}^n$ for some constant $c_{\alpha,\beta,\gamma,K}$, and write $L^m_{\rho,\delta}(\Omega)$ for the class of pseudo differential operators A of the form

$$Au(x) = (2\pi)^{-n} \int\int \exp[i\langle x - y, \xi\rangle] a(x, y, \xi) u(y)\, dy\, d\xi \tag{2}$$

with symbol $a(x, y, \xi) \in S^m_{\rho,\delta}(\Omega \times \Omega \times \mathbf{R}^n)$. In this note our aim will be to produce an example of a pseudo-local operator in $L^0_{1,1}$, which can displace the wave front set (WF) of distributions (for the definition of WF see Hörmander [2]).

Let us recall that a linear continuous map T of $C_0^\infty(\Omega)$ into $C^\infty(\Omega)$, and of $\mathcal{E}'(\Omega)$ into $\mathcal{D}'(\Omega)$, is said to be pseudo-local if

$$\text{sing supp } Tu \subset \text{sing supp } u \tag{3}$$

is valid for each $u \in \mathcal{E}'(\Omega)$ or, equivalently, the kernel $H \in \mathcal{D}'(\Omega \times \Omega)$ of T satisfies

$$\text{sing supp } H \subset \text{diagonal in } \Omega \times \Omega. \tag{4}$$

We shall say that T is strictly pseudo-local if

$$\text{WF } Au \subset \text{WF } u \tag{5}$$

for each $u \in \mathcal{E}'(\Omega)$. Since for every $f \in \mathcal{D}'(\Omega)$ the projection in Ω of WF $f \subset T^*(\Omega) \setminus 0$ is equal to sing supp f, (5) implies (3). Concerning pseudo differential operators, Hörmander has proved in [2]

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PROPOSITION 1. *Every* $A \in L^m_{\rho,\delta}(\Omega)$, $0 < \rho \leqslant 1$, $0 \leqslant \delta < 1$, *is strictly pseudo-local.*

Operators in the classes $L^m_{0,\delta}$, $0 \leqslant \delta \leqslant 1$, are not pseudo-local in general. For the other borderline case, $\delta = 1$, one can easily prove the following result (apply to (2) the argument in Nirenberg [**4**, p. 152], for example).

PROPOSITION 2. *Let A be in $L^m_{\rho,1}(\Omega)$, $0 < \rho \leqslant 1$. The kernel H of A satisfies* (4).

Therefore every $A \in L^m_{\rho,1}(\Omega)$, $0 < \rho \leqslant 1$, is a pseudo-local operator on Ω whenever it can be defined on $\mathcal{E}'(\Omega)$. Still, it need not be strictly pseudo-local as we show in

PROPOSITION 3. *Set $I = \{x, |x| < 2\pi\} \subset \mathbf{R}$ and write $T^*(I) \setminus 0 = \Lambda_+ \cup \Lambda_-$, $\Lambda_+ = I \times \mathbf{R}_+$, $\Lambda_- = I \times \mathbf{R}_-$. There exists $\Theta \in L^0_{1,1}(I)$ such that*:

(i) Θ: $C_0^\infty(I) \to C^\infty(I)$ *extends to a continuous linear map of $\mathcal{E}'(I)$ into $\mathcal{D}'(I)$.*

(ii) Θ *is continuous from $L^2_{\mathrm{comp}}(I)$ to $L^2_{\mathrm{loc}}(I)$.*

(iii) *For a suitable $u \in L^2_{\mathrm{comp}}(I)$ we have*: WF $u \neq \varnothing$, WF $\Theta u \neq \varnothing$ *and* WF $u \subset \Lambda_+$, WF $\Theta u \subset \Lambda_-$.

In §2 we shall define the symbol of Θ and we shall prove (i), (ii) (an operator in $L^0_{1,1}$ need not be L^2-continuous, in general); the proof of (iii) will be given in §§3,4.

2. Definition of Θ and L^2-continuity. Take $\varphi \in C_0^\infty(\mathbf{R})$, $0 \leqslant \varphi(x) \leqslant 1$, with $\varphi(x) = 1$ for $|x| \leqslant 1/2$, $\varphi(x) = 0$ for $|x| \geqslant 1$. Define, for $j = 1, 2, \ldots$:

$$\varphi_j(\xi) = \begin{cases} \varphi(\log_2\xi - 2j - 1) & \text{if } \xi \geqslant 1/2, \\ 0 & \text{if } \xi < 1/2. \end{cases} \tag{6}$$

Observe that $\varphi_j \in C_0^\infty(\mathbf{R})$, $0 \leqslant \varphi_j(x) \leqslant 1$, and

$$\operatorname{supp} \varphi_j \subset J_j = \{\xi, 2^{2j} \leqslant \xi \leqslant 2^{2j+2}\}, \tag{7}$$

$$\varphi_j(\xi) = 1 \quad \textit{if } 2^{2j+1/2} \leqslant \xi \leqslant 2^{2j+3/2}. \tag{8}$$

Note also that

$$|D_\xi^\gamma \varphi_j(\xi)| \leqslant c_\gamma (1 + |\xi|)^{-\gamma}, \tag{9}$$

with constants c_γ which do not depend on j; (9) is readily verified by induction on γ. Now set:

$$p(x, \xi) = \sum_{j=1}^{\infty} \exp[-i 2^{2j+1} 3x] \varphi_j(\xi), \tag{10}$$

and take $q \in C_0^\infty(\mathbf{R})$, $q(x) = 1$ for $|x| \leqslant 3\pi/2$, $q(x) = 0$ for $|x| \geqslant 5\pi/3$. Our symbol will be defined by

$$\vartheta(x, y, \xi) = q(x) q(y) p(x, \xi). \tag{11}$$

Since $2^{2j} < \xi$ for $\xi \in \operatorname{supp} \varphi_j$, we have from (9)

$$\left|D_x^\alpha D_\xi^\gamma \left\{\exp[-i2^{2j+1}3x]\varphi_j(\xi)\right\}\right| \leqslant 6^\alpha c_\gamma (1+|\xi|)^{\alpha-\gamma},$$

and, by observing that all terms in the sum (10) have disjoint supports, we can easily conclude that $\vartheta(x, y, \xi) \in S^0_{1,1}(I \times I \times \mathbf{R})$, with $I = \{x, |x| < 2\pi\}$.

The symbol $\vartheta(x, y, \xi)$ has compact support in the x, y variables and the corresponding operator $\Theta \in L^0_{1,1}(I)$ maps $C_0^\infty(I)$ into $C_0^\infty(I)$. In fact (11) and (2) (with $n = 1$, $a = \vartheta$) give:

$$\Theta u(x) = q(x)p(x, D)(qu)(x),$$

where for $f \in \mathcal{S}(\mathbf{R})$:

$$p(x, D)f(x) = (2\pi)^{-1}\int \exp[ix\xi]p(x, \xi)\hat{f}(\xi)\,d\xi. \tag{12}$$

We shall check that Θ is continuous from $L^2_{\text{comp}}(I)$ to $L^2_{\text{loc}}(I)$ by proving that $\|p(x, D)f\|^2 \leqslant \|f\|^2$ for every $f \in \mathcal{S}(\mathbf{R})$. Actually it will be sufficient to see that

$$\|[p(x, D)f]^\wedge\|^2 \leqslant \|\hat{f}\|^2 \tag{13}$$

for each f with $\hat{f}(\xi)$ compactly supported. In particular, assume $\operatorname{supp} \hat{f} \subset J = \{\xi, |\xi| < 2^{2m+2}\}$, with m sufficiently large; noting χ_j the characteristic function of J_j in (7) we can write

$$\sum_{j=1}^m \|\chi_j \hat{f}\|^2 \leqslant \|\hat{f}\|^2. \tag{14}$$

On the other hand, from (10), (12) it follows

$$p(x, D)f(x) = (2\pi)^{-1}\sum_{j=1}^m \exp[-i2^{2j+1}3x]\int \exp[ix\xi](\varphi_j \hat{f})(\xi)\,d\xi$$

and therefore

$$[p(x, D)f]^\wedge(\xi) = \sum_{j=1}^m (\varphi_j \hat{f})(\xi + 2^{2j+1}3). \tag{15}$$

In view of (7), all terms in the sum in (15) have disjoint supports. Then:

$$\|[p(x, D)f]^\wedge\|^2 = \sum_{j=1}^m \|\varphi_j \hat{f}\|^2 \leqslant \sum_{j=1}^m \|\chi_j \hat{f}\|^2,$$

which implies (13) in virtue of (14). By means of a repetition of the argument, we can also check easily that Θ is bounded on $H^s_{\text{comp}}(I)$, for every real s, and it extends to a continuous map of $\mathcal{E}'(I)$ into $\mathcal{E}'(I)$.

3. Θ is not strictly pseudo-local. We begin by defining the test function $u \in L^2_{\text{comp}}(I)$. Set

$$\mu(x) = \sum_{j=1}^\infty 2^{-2j}\exp[i2^{2j+1}x], \tag{16}$$

$$\nu(x) = \mu(-2x) = \sum_{j=1}^{\infty} 2^{-2j}\exp[-i2^{2j+2}x]. \tag{17}$$

Observe that $\mu(x)$, $\nu(x)$ are periodic with period 2π in $L^2_{\text{loc}}(R)$ and they satisfy:

$$\text{sing supp}\,\mu = \text{sing supp}\,\nu = \mathbf{R}. \tag{18}$$

(18) can be deduced from a classical result on lacunary Fourier series; otherwise, to prove (iii) in Proposition we shall only need

$$\text{sing supp}\,\mu \cap I_0 \neq \varnothing, \qquad \text{sing supp}\,\nu \cap I_0 \neq \varnothing \tag{19}$$

where

$$I_0 = \{x, |x| \leqslant \pi\}, \tag{20}$$

which is obvious, since the first order derivatives of the sums (16), (17) cannot be continuous functions in $\mathbf{R}$. Now take $\tau \in C_0^\infty(\mathbf{R})$, $\tau(x) \neq 0$ for $x \in I_0$ in (20), $\tau(x) = 0$ for $|x| \geqslant 3\pi/2$, and define

$$u(x) = \tau(x)\mu(x), \qquad v(x) = \tau(x)\nu(x). \tag{21}$$

We have $u \in L^2_{\text{comp}}(I)$, $v \in L^2_{\text{comp}}(I)$ and, in virtue of (19), WF $u \neq \varnothing$, WF $v \neq \varnothing$. In §4 we shall see that

$$\Theta u - v \in C_0^\infty(I). \tag{22}$$

Therefore Proposition 3 will follow from

$$\text{WF}\,u \subset \Lambda_+, \qquad \text{WF}\,v \subset \Lambda_-, \tag{23}$$

which we shall prove by using this equivalent definition of WF.

PROPOSITION 4 (HÖRMANDER [3, p. 127]). $(x_0, \xi_0) \notin \text{WF} f$ *if and only if one can find* $f_1 \in \mathcal{E}'$ *equal to* f *in a neighborhood of* x_0 *and with* $\hat{f}_1(\xi) = O(|\xi|^{-N})$ *for every* N *in a conic neighborhood of* ξ_0 *independent of* N.

In view of Proposition 4, to check WF $u \cap \Lambda_- = \varnothing$, WF $v \cap \Lambda_+ = \varnothing$, it will be sufficient to show that for every N there exists $c_N > 0$ such that

$$|(\tau\mu)\hat{}\,(\xi)| \leqslant c_N(-\xi)^{-N} \quad \text{for } \xi < 0, \tag{24}$$

$$|(\tau\nu)\hat{}\,(\xi)| \leqslant c_N\xi^{-N} \quad \text{for } \xi > 0. \tag{25}$$

First observe that in virtue of (16), (17)

$$(\tau\mu)\hat{}\,(\xi) = \sum_{j=1}^{\infty} 2^{-2j}\hat{\tau}(\xi - 2^{2j+1}), \tag{26}$$

$$(\tau\nu)\hat{}\,(\xi) = \sum_{j=1}^{\infty} 2^{-2j}\hat{\tau}(\xi + 2^{2j+2}). \tag{27}$$

Since $\hat{\tau} \in \mathcal{S}(\mathbf{R})$, there exists a constant c'_N such that

$$|\hat{\tau}(\xi)| \leqslant c'_N|\xi|^{-N} \tag{28}$$

and we can estimate

$$|\hat{\tau}(\xi - 2^{2j+1})| \leqslant c'_N(-\xi)^{-N} \quad \text{for } \xi < 0, \tag{29}$$

$$|\hat{\tau}(\xi + 2^{2j+2})| \leqslant c'_N\xi^{-N} \quad \text{for } \xi > 0. \tag{30}$$

Applying (29), (30) in (26), (27) we obtain (24), (25) with $c_N = c'_N/3$. The inclusions (23) are proved.

Observe that, using (18), we can deduce from (21): sing supp $u =$ sing supp $v =$ supp τ, and from (23): WF $u =$ supp $\tau \times \mathbf{R}_+$, WF $v =$ supp $\tau \times \mathbf{R}_-$. Let us point out also that, generalizing the definition in (10) and using test functions of the same type in $\mathbf{R}^n$, $n > 1$, one can easily construct examples of operators in $L^m_{1,1}(\Omega)$, $\Omega \subset \mathbf{R}^n$, with arbitrary m and Ω, which move the WF from a given ray to any other fixed ray or cone in the fibers of $T^*(\Omega)$.

4. Proof of (22). Since $q(x) = 1$ for $x \in$ supp τ, we have

$$\Theta u(x) = q(x)p(x, D)(\tau\mu)(x). \tag{31}$$

Therefore in view of (12), (26) $\Theta u(x)$ can be expressed by means of the Fourier integral

$$\Theta u(x) = q(x)(2\pi)^{-1}\int \exp[ix\xi]\chi(x, \xi)\,d\xi, \tag{32}$$

where $\chi \in C^\infty(I \times \mathbf{R})$ is given by

$$\chi(x, \xi) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \lambda_{j,k}(x, \xi) \tag{33}$$

with

$$\lambda_{j,k}(x, \xi) = 2^{-2k}\exp[-i2^{2j+1}3x]\varphi_j(\xi)\hat{\tau}(\xi - 2^{2k+1}). \tag{34}$$

Let us define χ_1, χ_2 in $C^\infty(I \times \mathbf{R})$,

$$\chi_1 = \sum_{j=1}^{\infty} \lambda_{j,j}, \qquad \chi_2 = \sum_{j=1}^{\infty} \sum_{\substack{k=1 \\ k \neq j}}^{\infty} \lambda_{j,k}, \tag{35}$$

and split consequently $\Theta u(x)$ into the sum of the two terms

$$v_t(x) = q(x)(2\pi)^{-1}\int \exp[ix\xi]\chi_t(x, \xi)\,d\xi, \qquad t = 1, 2. \tag{36}$$

The function v_2 is in $C_0^\infty(I)$, as we see immediately by differentiating (36) under the sign of integration and using the estimates

$$|D_x^\alpha \chi_2(x, \xi)| \leqslant K_{\alpha,N}(1 + |\xi|)^{-N}, \tag{37}$$

which will be proved at the end of this section. So, to obtain (22), it suffices to show $v_1 - v \in C_0^\infty(I)$. Writing out v_1 in (36) as a sum of integrals and applying to each term the transformation $\eta = \xi - 2^{2j+1}3$, we get

$$v_1(x) = q(x)(2\pi)^{-1}\int \exp[ix\eta] \sum_{j=1}^{\infty} 2^{-2j}\varphi_j(\eta + 2^{2j+1}3)\hat{\tau}(\eta + 2^{2j+2})\,d\eta.$$

On the other hand, since $q(x) = 1$ for $x \in$ supp v, in view of (21), (27) we can write

$$v(x) = q(x)(2\pi)^{-1}\int \exp[ix\eta] \sum_{j=1}^{\infty} 2^{-2j}\hat{\tau}(\eta + 2^{2j+2})\, d\eta$$

and we conclude

$$v_1(x) - v(x) = q(x)(2\pi)^{-1}\int \exp[ix\eta]\sigma(\eta)\, d\eta, \tag{38}$$

where $\sigma(\eta) \in C^\infty(\mathbf{R})$ is given by

$$\sigma(\eta) = \sum_{j=1}^{\infty} \sigma_j(\eta), \tag{39}$$

with

$$\sigma_j(\eta) = 2^{-2j}\hat{\tau}(\eta + 2^{2j+2})\left[\varphi_j(\eta + 2^{2j+1}3) - 1\right]. \tag{40}$$

We shall prove that for every N there exists C_N such that

$$|\sigma(\eta)|\, |\eta|^N \leqslant C_N. \tag{41}$$

For $\eta > 0$ the estimate is an easy consequence of (30). Take $\eta < 0$ and assume $-2^{2k+3} \leqslant \eta \leqslant -2^{2k+1}$. In view of (28) we have

$$\left|\hat{\tau}(\eta + 2^{2j+2})\right| \leqslant c'_N\left|\eta + 2^{2j+2}\right|^{-N}. \tag{42}$$

Then, noting that for $j \neq k$ it is $|\eta + 2^{2j+2}| \geqslant 2^{2k} \geqslant |\eta|/8$, we obtain from (40):

$$|\sigma_j(\eta)|\, |\eta|^N \leqslant 8^N c'_N 2^{-2j}, \quad \text{for } j \neq k. \tag{43}$$

On the other hand, observing that in virtue of (8) $\varphi_k(\eta + 2^{2k+1}3) = 1$ for $-2^{2k+2}21/20 \leqslant \eta \leqslant -2^{2k+2}19/20$, we deduce $|\eta + 2^{2k+2}| \geqslant 2^{2k}/10 \geqslant |\eta|/80$ for $\eta \in \operatorname{supp} \sigma_k$. It follows

$$|\sigma_k(\eta)|\, |\eta|^N \leqslant 80^N c'_N 2^{-2k}. \tag{44}$$

Therefore, in view of (43), (44), (39), the estimate (41) is satisfied also for negative η and (38) shows that $v_1 - v \in C_0^\infty(I)$. It remains to prove (37). To this end we observe that

$$|D_x^\alpha \chi_2(x, \xi)| \leqslant 6^\alpha |\xi|^\alpha \sum_{k=1}^{\infty} 2^{-2k}|\hat{\tau}(\xi - 2^{2k+1})|\Phi_k(\xi),$$

with

$$\Phi_k(\xi) = \sum_{\substack{j=1 \\ j \neq k}}^{\infty} \varphi_j(\xi).$$

Since $0 \leqslant \Phi_k(\xi) \leqslant 1$, and $\Phi_k(\xi) = 0$ for $2^{2k} \leqslant \xi \leqslant 2^{2k+2}$, we can repeat the argument which we used to estimate $\sigma(\eta)$; (37) follows easily, and this completes the proof of (22).

5. A remark on hypoellipticity. It has been observed recently that certain quasi-elliptic degenerate linear partial differential operators have left parametrices in the classes $L^m_{1/2,1}$ (see for example the study of the Kannai operator $\partial_x + x\partial_y^2$ in Beals [**1**, Examples 10.5 and 11.18]). These parametrices are certainly pseudo-local but they cannot be expected to be strictly pseudo-local; actually, though hypoelliptic, the corresponding operators are not strictly hypoelliptic in the sense of Hörmander [**3**, p. 151], i.e.: they decrease the WF (for $\partial_x + \partial_y^2$, this can be directly tested on suitable distributions).

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