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EXTREME INVARIANT POSITIVE OPERATORS ON L_p-SPACES

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ABSTRACT. Let $(X, \mathfrak{A}, \mu)$ and $(Y, \mathfrak{B}, \nu)$ be finite positive measure spaces. In this note we present characterizations of the extreme points of the convex set of all positive linear operators T: $L_p(\mu) \to L_q(\nu)$ with $T\mathbf{1}_X = \mathbf{1}_Y$ which are invariant with respect to a semigroup of positive constant preserving contractions on $L_p(\mu)$, $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$.

Introduction. Let $(X, \mathfrak{A}, \mu)$ and $(Y, \mathfrak{B}, \nu)$ be finite positive measure spaces and let $K[L_p(\mu), L_q(\nu)]$ denote the convex set of all positive linear operators T: $L_p(\mu) \to L_q(\nu)$ with $T\mathbf{1}_X = \mathbf{1}_Y$ for $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. It is known that an operator T in $K[L_p(\mu), L_q(\nu)]$ is an extreme point of this set if and only if T is a lattice homomorphism [**6**, III.9.2]. Further characterizations of the extreme points of $K[L_\infty(\mu), L_\infty(\nu)]$ as operators which are multiplicative or which carry characteristic functions into characteristic functions are given by Phelps [**5**, 2.2]. We will characterize the extreme points of the set $K[L_p(\mu), L_q(\nu)]_G$ of all operators in $K[L_p(\mu), L_q(\nu)]$ which are invariant with respect to a semigroup G of positive constant preserving contractions on $L_p(\mu)$ for $p < \infty$. These characterizations, including generalizations of the above mentioned results, are in part similar to those of the extreme points of $K[C(X), C(Y)]_G$ for compact spaces X and Y, which have been stated by Converse, Namioka and Phelps [**2**, 5.3].

1. Preliminaries. Throughout suppose that $(X, \mathfrak{A}, \mu)$ and $(Y, \mathfrak{B}, \nu)$ are finite positive measure spaces. $\mathbf{1}_A$ stands for the characteristic function of A. The convex set of all positive linear operators T: $L_p(\mu) \to L_q(\nu)$ with $T\mathbf{1}_X = \mathbf{1}_Y$ is denoted by $K[L_p(\mu), L_q(\nu)]$ for $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. Let G be a subsemigroup of $K[L_p(\mu), L_p(\mu)]$. A linear operator T: $L_p(\mu) \to L_q(\nu)$ is called invariant if $TV = T$ for all $V \in G$. $K[L_p(\mu), L_q(\nu)]_G$ denotes the convex set of all invariant elements in $K[L_p(\mu), L_q(\nu)]$. Furthermore, we denote by D the linear hull of the set $\{Vf - f\colon f \in L_p(\mu),\ V \in G\}$ and by F the fixed space of G, i.e. $F = \{f \in L_p(\mu)\colon Vf = f \text{ for all } V \in G\}$.

The key for the characterizations of extreme points is the following fact. If G is a contractive semigroup, i.e. $\sup\{\|V\|\colon V \in G\} \leqslant 1$ and $p < \infty$, then the semigroup $\mathrm{co}(G)^-$ has a zero element, where $\mathrm{co}(G)^-$ denotes the closed convex hull of G in the space of all continuous linear operators on $L_p(\mu)$,

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endowed with the topology of simple convergence [**4**, 1.4 and 2.3]. The (unique) zero element P of $\mathrm{co}(G)^-$ is a positive contractive projection onto F with $P\mathbf{1}_X = \mathbf{1}_X$ (cf. [**6**, III.7.2]). Furthermore, an operator $T \in K[L_p(\mu), L_q(\nu)]$ is invariant if and only if $TP = T$. This follows from the continuity of T (cf. [**6**, II.5.3]).

2. Extreme invariant operators. The following characterization, which holds without any further hypotheses on G and p, is a special case of [**3**, Theorem 5].

THEOREM 1. *Suppose $T \in K[L_p(\mu), L_q(\nu)]_G$. Then T is an extreme point of $K[L_p(\mu), L_q(\nu)]_G$ if and only if* $\inf\{T(|f - h|): h \in R\mathbf{1}_X + D\} = 0$ *for each* $f \in L_p(\mu)$.

Before we can formulate the main result, we need the following information.

LEMMA. *If G is a contractive semigroup, $p < \infty$, and P is the zero element of $\mathrm{co}(G)^-$, then $F = L_p(\mu|\mathfrak{A}_0)$ and P is the $\mathfrak{A}_0$-conditional expectation with $\mathfrak{A}_0 = \{A \in \mathfrak{A}: \mathbf{1}_A \in F\}$.*

PROOF. Obviously F is a closed subspace of $L_p(\mu)$ with $\mathbf{1}_X \in F$. Furthermore, F is a sublattice. Let $f \in F$ and $V \in G$. Since $f^+ \geqslant f$ and $f^+ \geqslant 0$ we have $Vf^+ \geqslant Vf = f$ and $Vf^+ \geqslant 0$. Hence, $Vf^+ \geqslant f^+$ and this implies $Vf^+ = f^+$ because V is a contraction. The first assertion follows from the well-known characterization of closed sublattices of $L_p(\mu)$ which contain $\mathbf{1}_X$ (cf. [**6**, III.11.2]). In view of the above mentioned properties of P, the second assertion is a result of Ando [**1**, (proof of) Theorem 2].

Let $E_{\mathfrak{A}_0}$ denote the $\mathfrak{A}_0$-conditional expectation operator.

THEOREM 2. *Suppose $T \in K[L_p(\mu), L_q(\nu)]_G$ and $p < \infty$. If G is a contractive semigroup, then the following assertions are equivalent.*

(i) *T is an extreme point of $K[L_p(\mu), L_q(\nu)]_G$.*

(ii) $\inf\{T(|f - t\mathbf{1}_X|): t \in R\} = 0$ *for each $f \in F$.*

(iii) $T(E_{\mathfrak{A}_0}f \cdot h) = TfTh$ *for each $f \in L_p(\mu)$, $h \in L_\infty(\mu)$.*

(iv) $T(fh) = TfTh$ *for each $f \in F$, $h \in L_\infty(\mu|\mathfrak{A}_0)$.*

(v) *$T|F$ is a lattice homomorphism.*

(vi) *T carries $\mathfrak{A}_0$-measurable characteristic functions into characteristic functions.*

PROOF. The equivalence of (i) and (ii) follows from [**3**, Theorem 6].

(i) $\Rightarrow$ (iii). Clearly it is sufficient to prove that assertion (iii) holds for those $h \in L_\infty(\mu)$ such that $0 \leqslant h \leqslant \mathbf{1}_X$. Assuming $0 \leqslant h \leqslant \mathbf{1}_X$, we define a map T_0: $L_p(\mu) \to L_q(\nu)$ by $T_0 f := T(E_{\mathfrak{A}_0}f \cdot h) - TfTh$. Then T_0 is an invariant linear operator with $T_0\mathbf{1}_X = 0$. If $f \geqslant 0$, then $(T + T_0)f = Tf(\mathbf{1}_Y - Th) + T(E_{\mathfrak{A}_0}f \cdot h) \geqslant 0$ and $(T - T_0)f = T[E_{\mathfrak{A}_0}f \cdot (\mathbf{1}_X - h)] + TfTh \geqslant 0$. Thus $T \pm T_0 \in K[L_p(\mu), L_q(\nu)]_G$ holds. Since T is extreme, this implies $T_0 = 0$.

(iii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (v). First let $f \in L_\infty(\mu|\mathfrak{A}_0)$. The chain $(T|f|)^2 = T(|f|^2) = T(f^2) = (Tf)^2 = |Tf|^2$ shows that $|Tf| = T|f|$ holds. Since $L_\infty(\mu|\mathfrak{A}_0)$ is dense in F and furthermore, T and the lattice operations on $L_p(\mu)$ and $L_q(\nu)$ are continuous, this implies that $T|F$ is a lattice homomorphism.

(v) $\Rightarrow$ (vi). For $A \in \mathfrak{A}_0$ we obtain $T\mathbf{1}_A \wedge T\mathbf{1}_{A^c} = T(\mathbf{1}_A \wedge \mathbf{1}_{A^c}) = T0 = 0$. Hence, $T\mathbf{1}_A$ is a characteristic function.

(vi) $\Rightarrow$ (ii). Let $A \in \mathfrak{A}_0$. Since $T(|\mathbf{1}_A - t\mathbf{1}_X|) = |1 - t|T\mathbf{1}_A + |t|T\mathbf{1}_{A^c}$, it follows that

$$\inf\{T(|\mathbf{1}_A - t\mathbf{1}_X|): t \in R\} \leqslant T\mathbf{1}_A \wedge T\mathbf{1}_{A^c} = 0.$$

By virtue of the continuity of T it is readily verified that $\inf\{T(|f - t\mathbf{1}_X|): t \in R\} = 0$ is valid for each $f \in F$.

Corollary 1. *Suppose $T \in K[L_p(\mu), R]_G$. Under the above hypotheses on G and p, T is an extreme point of $K[L_p(\mu), R]_G$ if and only if $T\mathbf{1}_A \in \{0, 1\}$ for each $A \in \mathfrak{A}_0$.*

In the following corollary an application to conditional expectations is given.

Corollary 2. *Suppose that $\mathfrak{A}_1$ is a σ-subalgebra of $\mathfrak{A}_0$. Under the above hypotheses on G and p, the operator $E_{\mathfrak{A}_1}$ is an extreme point of $K[L_p(\mu), L_p(\mu)]_G$ if and only if for each $A \in \mathfrak{A}_0$ there exists $B \in \mathfrak{A}_1$ with $\mu(A \triangle B) = 0$.*

Proof. Obviously $E_{\mathfrak{A}_1} \in K[L_p(\mu), L_p(\mu)]_G$ holds.

The "if" part. Let $A \in \mathfrak{A}_0$. By assumption there exists $B \in \mathfrak{A}_1$ with $\mu(A \triangle B) = 0$ such that $E_{\mathfrak{A}_1}\mathbf{1}_A = E_{\mathfrak{A}_1}\mathbf{1}_B = \mathbf{1}_B$. The assertion follows from Theorem 2.

The "only if" part. Let $A \in \mathfrak{A}_0$. From Theorem 2 follows

$$\begin{aligned}\mu(A \cap C) &= \int E_{\mathfrak{A}_1}(\mathbf{1}_A\mathbf{1}_C)d\mu \\ &= \int E_{\mathfrak{A}_1}\mathbf{1}_A E_{\mathfrak{A}_1}\mathbf{1}_C d\mu = \int \mathbf{1}_C E_{\mathfrak{A}_1}\mathbf{1}_A d\mu\end{aligned}$$

for each $C \in \mathfrak{A}_0$. Hence, $E_{\mathfrak{A}_1}\mathbf{1}_A = \mathbf{1}_A$. This yields the assertion.

Remark. Let $G \subset K[L_\infty(\mu), L_\infty(\mu)]$. Then $V \in G$ can be extended to a positive linear contraction on $L_1(\mu)$ if (and only if) V is expectation invariant, i.e. $\int Vf\,d\mu = \int f\,d\mu$ for each $f \in L_\infty(\mu)$. Thus the preceding characterizations are valid for $p = \infty$ if G is a semigroup of expectation invariant operators.

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