PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 72, Number 2, November 1978

TWO RESULTS RELATING NILPOTENT SPACES AND COFIBRATIONS

ROBERT H. LEWIS

ABSTRACT. We first prove a Blakers-Massey Theorem for nilpotent spaces: If (X, A) is an n-connected, $n \geqslant 1$, pair of nilpotent spaces, then under suitable conditions the map $\pi_*(X, A) \to \pi_* X/A$ is an isomorphism in dimension $n + 1$ and an epimorphism in dimension $n + 2$. Next, we dualize the well-known fact that if the total space of a fibration is nilpotent, so is the fiber. Our dual theorem can be used to construct new examples of finite nilpotent CW complexes.

0. Introduction. A topological space X is said to be nilpotent if its fundamental group is nilpotent and if the homotopy groups $\pi_n X$, $n \geqslant 2$, are nilpotent modules over the group ring $Z\pi_1 X$. Dror [1] motivated the study of nilpotent spaces by proving that such spaces satisfy the Whitehead Theorem. Since then, many theorems previously established for simply connected spaces have been generalized to include nilpotent spaces. The book by Hilton, Mislin, and Roitberg [4] contains some of these results.

Because it is defined in terms of homotopy groups, nilpotency behaves well with respect to fibrations. For instance, a well-known theorem asserts that if the total space of a fibration is nilpotent, so is the fiber. The purpose of the present work is to establish two results connecting nilpotency to cofibrations. In §1 we prove a Blakers-Massey Theorem for pairs of nilpotent spaces. In §2 we assume that the total space of a cofibration is nilpotent and show that under certain conditions the cofiber is nilpotent. As a corollary of this theorem we have an easy way of constructing many new examples of nilpotent spaces which are finite CW complexes.

In discussing nilpotent modules we use the notation of Dror in [1]. Whenever convenient, "space" will mean path connected CW complex. We use $K'(G, n)$ to denote a Moore space: a simply connected CW complex with the single nonvanishing homology group G in dimension n. All tensor products are taken over the ring Z of integers.

1. A Blakers-Massey Theorem for nilpotent spaces.

DEFINITION. We say that a space is nilpotent up to dimension n if $\pi_1 X$ acts nilpotently on $\pi_k X$ for $k = 1, 2, \ldots, n$.

LEMMA 1. *Let X be nilpotent up to dimension n and Y be nilpotent up to*

Presented to the Society, January 4, 1978; received by the editors October 17, 1977 and, in revised form, March 8, 1978.

AMS (MOS) subject classifications (1970). Primary 55D30, 55D05, 55E35.

Key words and phrases. Nilpotent space, cofibration, Blakers-Massey Theorem, Moore space.

dimension $n+1$. If $f\colon X \to Y$ induces $f_*\colon H_iX \to H_iY$ isomorphic for $i \leqslant n$ and epimorphic for $i = n+1$, then $f_*\colon \pi_iX \to \pi_iY$ is isomorphic for $i \leqslant n$ and epimorphic for $i = n+1$ ($n \geqslant 1$).

PROOF. This was essentially proven by Gersten in [2], though he did not state it. It is a finite dimensional version of Dror's theorem in [1].

By a result of Stallings [6], we conclude that $f_*\colon \pi_1X \xrightarrow{\sim} \pi_1Y$. From the sequence

$$\pi_2X \to \pi_2Y \to \pi_2(Y, X) \to 0$$

and the Relative Hurewicz Theorem we deduce the surjectivity of $\pi_2X \to \pi_2Y$, thus establishing the case $n = 2$.

Next, assume inductively that $\pi_kX \to \pi_kY$ is isomorphic for $k \leqslant n-1$ and epimorphic for $k = n$. Consider

$$\pi_{n+1}X \to \pi_{n+1}Y \to \pi_{n+1}(Y, X) \to \pi_nX \to \pi_nY.$$

The nilpotency of the π ($= \pi_1X = \pi_1Y$)-modules $\pi_{n+1}Y$ and π_nX implies that of $\pi_{n+1}(Y, X)$. But by the Relative Hurewicz Theorem,

$$\pi_{n+1}(Y, X)/\Gamma_2\pi_{n+1}(Y, X) = H_{n+1}(Y, X) = 0.$$

This means that $\pi_{n+1}(Y, X)$ is a perfect π-module, and so is trivial. This establishes the injectivity of $\pi_nX \to \pi_nY$ and the surjectivity of $\pi_{n+1}X \to \pi_{n+1}Y$, as desired.

The lemma is used to prove the following Blakers-Massey Theorem for nilpotent spaces.

THEOREM 1. *Let (X, A) be an n-connected cofiber pair where A is nilpotent to dimension n and X is nilpotent to dimension $n+1$, $n \geqslant 1$. (If $n = 1$, assume also that $H_1A \to H_1X$ is an isomorphism.) If $H_1X \otimes H_{n+1}(X/A) = 0$ then the map $p\colon X \to X/A$ induces an isomorphism*

$$p_*\colon \pi_{n+1}(X, A) \to \pi_{n+1}(X/A)$$

and an epimorphism in dimension $n+2$.

PROOF. The proof is patterned after that of the standard Blakers-Massey Theorem given in Spanier [5].

We have the diagram

$$\begin{tikzcd}
\pi_k(X, A) \arrow[rr, "i_*"] \arrow[dr, "p_*"'] & & \pi_k(X \cup CA, CA) \arrow[dl, "\simeq"] \\
& \pi_k(X/A) &
\end{tikzcd}$$

so we may shift attention to i_*. Replace the inclusion $X \to X \cup CA$ with the equivalent fibration $F \to X' \xrightarrow{q} X \cup CA$ (the mapping-track construction).

Note that $H_1F \cong H_1X$ (even if $n = 1$, by the parenthetical hypothesis $H_1A \cong H_1X$). Standard spectral sequence techniques applied to the Serre spectral sequence of this fibration show that

$$q_*: H_k(X'', E) \to H_k(X \cup CA, CA)$$

is an isomorphism for $k \leqslant n + 2$ and an epimorphism for $k = n + 3$. "E" here is $q^{-1}(CA)$. The point is that the hypotheses are sufficient to force $E^2_{n+1,1} = 0$. The map i: $(X, A) \to (X \cup CA, CA)$ has a lift $\bar{i}$: $(X, A) \to (X', E)$. We have the diagram

$$\begin{tikzcd} H_k(X, A) \arrow[rr, "\bar{i}_*"] \arrow[dr, "i_*"', "\cong"] & & H_k(X', E) \arrow[dl, "q_*"] \\ & H_k(X \cup CA, CA) & \end{tikzcd}$$

Hence $\bar{i}_*$ is isomorphic for $k \leqslant n + 2$. From the Five-Lemma we conclude that $H_kA \to H_kE$ is isomorphic for $k \leqslant n + 1$. The space E is homotopy equivalent to F, which is nilpotent to dimension $n + 1$ because X' is nilpotent to dimension $n + 1$. From our Lemma 1, then, we conclude that the homomorphism $\pi_kA \to \pi_kE$ is an isomorphism for $k \leqslant n$ and an epimorphism when $k = n + 1$. By the Five-Lemma, $\pi_k(X, A) \to \pi_k(X', E)$ is isomorphic when $k \leqslant n + 1$ and epimorphic if $k = n + 2$. Use the natural isomorphism

$$\pi_*(X', E) \to \pi_*(X \cup CA, CA)$$

to complete the proof.

It is not harder to get more results by adding a few hypotheses. For instance, if

$$H_1X \otimes H_{n+1}(X/A) = 0, \quad H_1X \otimes H_{n+2}(X/A) = 0,$$

and $H_2F \otimes H_{n+1}(X/A) = 0$ ($H_2F = H_2X$ for large n), then a few more terms in the spectral sequence are killed off. Thus, if we assume that X and A are each one dimension more highly nilpotent, we can show that the map p_* is an isomorphism in dimension $n + 2$ and an epimorphism in dimension $n + 3$.

A different Blakers-Massey Theorem for nilpotent spaces is Theorem 4.4 of the paper by Hilton and Roitberg [7].

2. The nilpotency of X/A when X is nilpotent. It is well known that in a fibration the fiber is nilpotent whenever the total space is nilpotent. The purpose of this section is to prove the following theorem:

THEOREM 2. *Let (X, A) be a cofibered pair with cofiber C. Then C is nilpotent iff X is nilpotent and $Z(\pi_1C) \otimes H_iA$ is a nilpotent π_1C-module for all $i > 0$.*

PROOF. We use the fact that a space is nilpotent iff its fundamental group is nilpotent and acts nilpotently on the homology of its universal cover (see Hilton, et al. [4]). We need to describe the universal cover of C.

Let $\tilde{X}_A$ denote the cover of X corresponding to the normal subgroup of $\pi_1 X$ generated by elements of $\pi_1 A$. The inclusion i: $A \to X$ has a lift to $\tilde{X}_A$; in fact there is one lift i_α for each element of the fiber F of $\tilde{X}_A \to X$. We have the following diagram, where $\tilde{C}$ is the indicated pushout:

$$\begin{tikzcd}
\bigoplus_{\alpha \in \pi_0 F} A \arrow[r] \arrow[d] & \tilde{X}_A \arrow[d] \arrow[dr] & \\
\bigoplus_{\alpha \in \pi_0 F} CA \arrow[r] \arrow[dr] & \tilde{C} \arrow[d, dashed] & X \arrow[dl] \\
& X \cup CA = C &
\end{tikzcd}$$

The induced map is obviously a universal covering. We also have a cofibration $\tilde{X}_A \to \tilde{C} \to V\Sigma A$, in which the cofiber has as many copies of ΣA as there are elements of $\pi_0 F = \pi_1 X / \langle \pi_1 A \rangle = \pi_1 C$. To complete the proof, we will show that in the sequence of $\pi_1 C$-modules

$$\cdots \to H_i \tilde{X}_A \to H_i \tilde{C} \to H_i V\Sigma A \to \cdots$$

$H_* \tilde{X}_A$ and $H_* V\Sigma A$ are nilpotent modules.

We will first show that $H_* \tilde{X}_A$ is a nilpotent $\pi_1 C$-module. $\pi_1 X$ acts on $\tilde{X}$, the universal cover of X, via deck transformations. This action naturally induces an action of $\pi_1 X$ on $\tilde{X}_A$ which is the same as the $\pi_1 C = \pi_1 X / \langle \pi_1 A \rangle$ action. Clearly, $H_* \tilde{X}_A$ is a nilpotent $\pi_1 C$-module iff it is a nilpotent $\pi_1 X$-module, so we will prove the latter.

Given an element α of $\pi_1 X$ let h: $\tilde{X}_A \to \tilde{X}_A$ be the associated deck transformation. This h is not base point preserving, but it can be homotoped to a map h' which does preserve base point. It turns out that h'_*: $\pi_1 \tilde{X}_A \to \pi_1 \tilde{X}_A$ is conjugation by α. Then h' can be lifted to a map (which we also call h') h': $\tilde{X} \to \tilde{X}$ which is base point preserving and freely homotopic to the deck transformation of $\tilde{X}$ associated with α. Also, there is an associated based map

$$h' \colon K(\langle \pi_1 A \rangle, 1) \to K(\langle \pi_1 A \rangle, 1)$$

inducing conjugation by α on π_1. In this way $\pi_1 X$ acts on the fibration

$$\begin{tikzcd}
\tilde{X} \arrow[r] \arrow[d, "h'"] & \tilde{X}_A \arrow[r] \arrow[d, "h'"] & K(\langle \pi_1 A \rangle, 1) \arrow[d, "h'"] \\
\tilde{X} \arrow[r] & \tilde{X}_A \arrow[r] & K(\langle \pi_1 A \rangle, 1)
\end{tikzcd}$$

(each square based homotopy commuting) in such a way that the induced actions on $H_* \tilde{X}$ and $H_* \tilde{X}_A$ are the natural ones.

It follows that $\pi_1 X$ acts on the E^2 term of the Serre spectral sequence of the fibration $\tilde{X}_A \to K(\langle \pi_1 A \rangle, 1)$ and that the boundary maps d^r are $\pi_1 X$-module maps. Note that the action on

$$E^2_{st} = H_s\big(K(\langle \pi_1 A\rangle, 1); H_t\tilde{X}\big)$$

involves both the space $K(\langle \pi_1 A\rangle, 1)$ and the coefficient group $H_t\tilde{X}$. For this reason, it is important that the maps h' be base point preserving.

Since $\pi_1 X$ is a nilpotent group it acts nilpotently on $\langle \pi_1 A\rangle$ and hence on $H_s(K(\langle \pi_1 A\rangle, 1); G)$ for any trivial $\langle \pi_1 A\rangle$-module G. (The proof is similar to that of Lemma 2.17, p. 69 of **[4]**). Let

$$0 \subset \Gamma_n \subset \Gamma_{n-1} \subset \cdots \subset \Gamma_2 \subset H_t\tilde{X}$$

be the $\pi_1 X$-lower central series of $H_t\tilde{X}$. The inclusions $\Gamma_i \subset \Gamma_{i-1}$ are also $\langle \pi_1 A\rangle$-maps. In the long exact $\langle \pi_1 A\rangle$-homology sequence associated with $0 \to \Gamma_n \to \Gamma_{n-1} \to \Gamma_{n-1}/\Gamma_n \to 0$ (the maps of which are also $\pi_1 X$-maps),

$$H_s\big(K(\langle \pi_1 A\rangle, 1); \Gamma_n\big) \quad \text{and} \quad H_s\big(K(\langle \pi_1 A\rangle, 1); \Gamma_{n-1}/\Gamma_n\big)$$

are nilpotent $\pi_1 X$-modules, by earlier reasoning. Hence $H_s(K(\langle \pi_1 A\rangle, 1); \Gamma_{n-1})$ is nilpotent. Proceed inductively to complete the proof that E^2_{st} is a nilpotent $\pi_1 X$-module.

To complete the proof of the theorem we need only observe that, as a $\pi_1 C$-module, $\tilde{H}_i V\Sigma A = Z(\pi_1 C) \otimes \tilde{H}_{i-1}A$.

COROLLARY 1. *If the cone on a Moore space $K'(G, n)$, $n \geqslant 2$ is attached to a nilpotent space X by a map $K'(G, n) \to X$, the resulting space is nilpotent iff $Z(\pi_1 X) \otimes G$ is a nilpotent $\pi_1 X$-module.*

COROLLARY 2. *In a cofibered pair (X, A), if X is nilpotent and A is acyclic then X/A is nilpotent and $X \to X/A$ is a homotopy equivalence.*

Reading Corollary 1 leads to the question: for what combinations of Abelian groups G and groups π is it true that $Z\pi \otimes G$ is a nilpotent π-module?

LEMMA. *If $\pi = Z_p$ for p a prime, then $\Gamma_{np+1}(Z\pi) \subset p^n Z\pi$, $n \geqslant 1$.*

PROOF. Let x denote the generator of Z_p, so that an element of Z_p is represented by x^r for $0 \leqslant r \leqslant p-1$. $\Gamma_2(Z\pi)$ is generated by elements of the form $(x^r - 1)c$, where $c \in Z\pi$. By factoring $(x-1)$ out of $(x^r - 1)$ we see that $\Gamma_2(Z\pi) \subset (x-1)Z\pi$. Inductively, $\Gamma_n(Z\pi) \subset (x-1)^{n-1}Z\pi$. Hence $\Gamma_{p+1}(Z\pi) \subset (x-1)^p Z\pi$. But since p is a prime it is easy to see that $(x-1)^p$ is a multiple of p. Thus $\Gamma_{p+1}(Z\pi) \subset pZ\pi$. By induction, we have the formula $\Gamma_{np+1}(Z\pi) \subset p^n Z\pi$.

PROPOSITION. *If $\pi = Z_p$, p prime, and G is a p-group, then $Z\pi \otimes G$ is a nilpotent π-module.*

PROOF. This follows from the lemma and the fact that $\Gamma_n(Z\pi \otimes G) \subset \Gamma_n(Z\pi) \otimes G$.

We see then that it does happen reasonably often that $Z\pi \otimes G$ is a nilpotent π-module. Hence Theorem 2 and its Corollaries are not vacuous. Corollary 1 can be used to construct many new examples of nilpotent complexes having few cells. For instance, we may take X in Corollary 1 to be RP^n, n odd, and G to be any two-group.

For a general reference on the subject of calculating $\Gamma_n(Z\pi)$ see the paper by Gruenberg [3].

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DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NEW YORK 14853

Current address: Department of Mathematics, Lander College, Greenwood, South Carolina 29646