PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 72, Number 2, November 1978

A NOTE ON SEMITOPOLOGICAL PROPERTIES

S. GENE CROSSLEY

ABSTRACT. The strongly Hausdorff and Urysohn properties of a topological space are shown to be semitopological properties.

I. Introduction. Levine [6] defined a set A to be *semiopen* in a topological space if and only if there is an open set U so that $U \subset A \subset c(U)$ where $c(\ )$ denotes the closure in the topological space.

In [2], *semiclosed* sets, *semi-interior*, and *semiclosure* were defined in a manner analogous to the corresponding concepts of closed, interior, and closure. Then in [3] a property of topological spaces was defined to be a *semitopological property* if it was preserved by *semihomeomorphisms* (bijections so that the images of semiopen sets are semiopen and inverses of semiopen sets are semiopen). In [3] the first category, Hausdorff, separable, and connected properties of topological spaces were shown to be semitopological properties.

The new separation axioms (*semi*-T_0, *semi*-T_1, and *semi*-T_2) defined by Maheshwari and Prasad [7] are also semitopological properties, and Hamlett showed [5] that the property of a topological space being a Baire space is semitopological. In this note two additional separation axioms closely related to the Hausdorff separation axiom are shown to be semitopological properties.

The method of proof, in [3], used to show that the Hausdorff property and connectedness were semitopological properties hinged on the fact that if $[\tau]$ is the equivalence class of topologies on X which yield the same semiopen sets then there is a finest element of $[\tau]$, denoted by $F(\tau)$. Also, if $f\colon (X, \tau) \to (Y, \sigma)$ is a semihomeomorphism, then $f\colon (X, F(\tau)) \to (Y, F(\sigma))$ is a homeomorphism. A new characterization of $F(\tau)$ as $\{0 - N \mid 0 \in \tau$ and N is nowhere dense in $(X, \tau)\}$ was given in [1], and this characterization has simplified the proofs given in this paper and it could be used to simplify the proof given in [3] that the Hausdorff separation axiom is a semitopological property.

II. The strongly Hausdorff and Urysohn properties of topological spaces are semitopological properties. The following lemma gives a key part of the proof that the Hausdorff property is a semitopological property in a manner

Received by the editors February 20, 1978.
AMS (MOS) subject classifications (1970). Primary 54A05.
Key words and phrases. Semiopen sets, semitopological properties, strongly Hausdorff, Urysohn space.

significantly shorter than that given in [3].

LEMMA 1. *If $F(\tau)$ is a Hausdorff topology then τ is a Hausdorff topology.*

PROOF. The contrapositive will be proved. If τ is not a Hausdorff topology on X, then there are distinct points x and y in X so that for each pair of open sets $U \in \tau$ and $V \in \tau$ so that $x \in U$ and $y \in V$ it must be the case that $U \cap V \neq \varnothing$. Now if $N_1 \subset U$, and $N_2 \subset V$ are nowhere dense subsets in (X, τ) so that $x \notin N_1$ and $y \notin N_2$, then $x \in U - N_1$ and $y \in V - N_2$ and we have $(U - N_1) \cap (V - N_2) = (U \cap V) - (N_1 \cup N_2)$. Furthermore, since $U \cap V$ is a nonvoid open set and $N_1 \cup N_2$ is nowhere dense $U \cap V \not\subset N_1 \cup N_2$ so that $(U \cap V) - (N_1 \cup N_2) \neq \varnothing$.

Since all open sets in $F(\tau)$ which contain x are of the form $U - N$ where $U \in \tau$, $N \subset U$, N is a nowhere dense set in (X, τ) and $x \notin N$, we see that $F(\tau)$ is not a Hausdorff topology when τ is not.

Hajnal and Juhasz have defined [4] a Hausdorff topological space to be strongly Hausdorff if and only if for each infinite subset $A \subset X$ there is a sequence $\{U_n | n \in P\}$ (P is the set of positive integers) of pairwise disjoint open sets such that $A \cap U_n \neq \varnothing$ for each $n \in P$.

THEOREM 1. *If (X, σ) is a strongly Hausdorff space and $\sigma \in [\tau]$ then (X, τ) is strongly Hausdorff.*

PROOF. Since (X, σ) is strongly Hausdorff and $\sigma \subset F(\tau)$, $(X, F(\tau))$ is strongly Hausdorff [4]. Since $(X, F(\tau))$ is Hausdorff, (X, τ) is Hausdorff by Lemma 1. Now let $A \subset X$ be any infinite subset of X. Since $(X, F(\tau))$ is strongly Hausdorff there is a sequence of pairwise disjoint open sets $\{U_n | n \in P\}$ of elements of $F(\tau)$ such that $A \cup U_n \neq \varnothing$ for each $n \in P$. Now for each $n \in P$ there exists a set $V_n \in \tau$ and N_n a nowhere dense set in (X, τ) so that $N_n \subset V_n$ and $V_n - N_n = U_n$ so that $V_n = U_n \cup N_n$. If $i \in P$ and $j \in P$ and $i \neq j$, $U_i \cap U_j = \varnothing$, thus we have

$$\begin{aligned} V_i \cap V_j &= (U_i \cup N_i) \cap (U_j \cup N_j) \\ &= (N_i \cap U_j) \cup (U_i \cap N_j) \cup (N_i \cap N_j) \subset N_i \cup N_j. \end{aligned}$$

But $V_i \cap V_j$ is open in (X, τ) and $N_i \cup N_j$ is nowhere dense in (X, τ) so that $V_i \cap V_j = \varnothing$. Thus $\{V_i | i \in P\}$ is a sequence of mutually disjoint elements τ. Furthermore, we have

$$A \cap V_n \supset A \cap U_n \neq \varnothing \quad \text{for each } n \in P.$$

Consequently (X, τ) is strongly Hausdorff.

COROLLARY 1. *The property of being strongly Hausdorff is a semitopological property.*

PROOF. If f: $(X, \tau) \to (Y, \sigma)$ is a semihomeomorphism and (X, τ) is strongly Hausdorff, then by Theorem 1, $(X, F(\tau))$ is strongly Hausdorff. Since f:

$(X, F(\tau)) \to (Y, F(\sigma))$ is a homeomorphism [3], $(Y, F(\sigma))$ is strongly Hausdorff. Finally by Theorem 1, (Y, σ) is strongly Hausdorff.

A topological space (X, τ) is a *Urysohn space* if and only if, for each pair of points $x \in X$ and $y \in X$ there exist open sets U and V so that $x \in U, y \in V$, and $c(U) \cap c(V) = \varnothing$.

THEOREM 2. *If (X, σ) is a Urysohn space and $\sigma \in [\tau]$ then (X, τ) is Urysohn.*

PROOF. If (X, σ) is Urysohn, then since $\sigma \subset F(\tau)$ it follows that $(X, F(\tau))$ is a Urysohn space. Thus the proof will be complete if it can be shown that whenever the finest topology in $[\tau]$ is Urysohn then (X, τ) is Urysohn. Since $(X, F(\tau))$ is Urysohn, $(X, F(\tau))$ is Hausdorff so that (X, τ) is Hausdorff by Lemma 1.

If (X, τ) is not a Urysohn space then there exist distinct points $a \in X$ and $b \in X$ so that for no pair of sets $U \in \tau$ and $V \in \tau$ do we have $a \in U$, $b \in V$, $c(U) \cap c(V) = \varnothing$. Now if $S \in \tau$ and $T \in \tau$ so that $a \in S$ and $b \in T$ and $S \cap T = \varnothing$ we must still have $c(S) \cap c(T) \neq \varnothing$. If N_1 and N_2 are nowhere dense in (X, τ) so that $a \notin N_1$ and $b \notin N_2$ then $(S - N_1) \in F(\tau)$, $(T - N_2) \in F(\tau)$, $a \in (S - N_1)$ and $b \in (T - N_2)$, but

$$c^*(S - N_1) \cap c^*(T - N_2) \subset c(S) \cap c(T)$$

where $c(\ )$ denotes closure in (X, τ) and $c^*(\ )$ denotes the closure in $(X, F(\tau))$.

Now if $q \in c(S) \cap c(T)$, let $W \in F(\sigma)$ so that $q \in W$. There is a set N_3, disjoint from W, and nowhere dense in (X, τ) so that $W \cup N_3 \in \tau$. Since $q \in c(S)$, we have $S \cap (W \cup N_3) \neq \varnothing$ and $S \cap (W \cup N_3) \in \sigma$. We have

$$(S - N_1) \cap (W) = (S \cap (W \cup N_3)) - (N_1 \cup N_3).$$

Notice that since $N_1 \cup N_3$ is nowhere dense in (X, τ) and $(S \cap (W \cup N_3)) \in \tau$, $(S \cap (W \cup N_3)) - (N_1 \cup N_3)$ is not empty. Thus $q \in c^*(S - N_1)$. By a similar argument $q \in c^*(T - N_2)$. Consequently, we have $c^*(S - N_1) \cap c^*(T - N_2) = c(S) \cap c(T)$.

Thus, we see that if there are open sets U and V in $F(\tau)$ so that $a \in U$, $b \in V$ and $c^*(U) \cap c^*(V) = \varnothing$ they cannot be obtained by taking disjoint neighborhoods of a and b in (X, τ) and subtracting nowhere dense sets. On the other hand, the proof of Lemma 1 shows that disjoint elements of $F(\tau)$ cannot be obtained from nondisjoint elements of τ by subtracting nowhere dense sets.

Since all open sets in $F(\tau)$ are of the form $0 - N$ where $0 \in \tau$ and N is nowhere dense in (X, τ) [1], we see that if (X, τ) is not Urysohn, neither is $(X, F(\tau))$, and the proof of Theorem 2 is complete.

The proof of the following corollary is essentially the same as that of Corollary 1.

COROLLARY 2. *The property of being a Urysohn space is a semitopological property.*

References

1. S. Gene Crossley, *A note on semitopological classes*, Proc. Amer. Math. Soc. **43** (1974), pp. 416–420.

2. S. Gene Crossley and S. K. Hildebrand, *Semi-closure*, Texas J. Sci. **22** (1971), pp. 99–112.

3. ______, *Semi-topological properties*, Fund. Math. **74** (1972), pp. 233–254.

4. A. Hajnal and I. Juhasz, *Some remarks on a property of topological cardinal functions*, Acta. Math. Acad. Sci. Hungar. **20** (1969), pp. 25–37.

5. T. R. Hamlett, *The property of being a Baire space is semi-topological*, Math. Chronicle **5** (1977), pp. 166–167.

6. N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), pp. 36–41.

7. S. N. Maheshwari and R. Prasad, *Some new separation axioms*, Ann. Soc. Sci. Bruxelles **89** (1975), pp. 395–402.

8. Takashi Noiri, *On semi-T_2 spaces*, Ann. Soc. Sci. Bruxelles **90** (1976), pp. 215–220.

9. J. R. Porter, *Strongly Hausdorff spaces*, Acta. Math. Acad. Sci. Hungar. **25** (1974), pp. 245–248.

Department of Mathematics, Southwest Texas State University, San Marcos, Texas 78666