PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 72, Number 3, December 1978

A KOROVKIN-TYPE THEOREM IN LOCALLY CONVEX M-SPACES

HANS O. FLÖSSER

ABSTRACT. Let E be a locally convex M-space, $\varnothing \neq M$ a subset. The universal Korovkin-closure of M as well as the sequentially or stationary defined Korovkin-closures coincide with the space of M-harmonic elements and with the uniqueness closure of M.

1. The Theorem. Let E, F denote locally convex vector lattices (**[6]**, but not necessarily separated); $L(E, F)_+$ is the cone of continuous positive linear operators from E into F and $V(E, F)$ the set of all continuous linear lattice homomorphisms from E into F. We write E'_+ for $L(E, \mathbf{R})_+$ and $V(E)$ for $V(E, \mathbf{R})$. For $\varnothing \neq M \subset E$ the universal Korovkin closure $K(M)$ (**[9]**, **[10]**) is defined by $e \in K(M)$

$$\left\{\begin{array}{l}\text{iff for any locally convex vector lattice } F\\ \text{and for any equicontinuous net } T_\alpha \in L(E, F)_+\\ \text{and for any } S \in V(E, F) \text{ the relation}\\ \lim\limits_\alpha T_\alpha f = Sf \text{ for all } f \in M \text{ implies } \lim\limits_\alpha T_\alpha e = Se.\end{array}\right. \tag{$*$}$$

Let $K_\sigma(M)$ and $K_0(M)$ denote the set of all $e \in E$ that satisfy ($*$), when in ($*$) the word "net" is replaced by "sequence" and "stationary sequence", respectively.

$L(M)$ is the closed linear hull of M. Let $\hat{M}$ be the set of all finite infima of elements in $L(M)$, i.e.

$$\hat{M} = \{\textstyle\bigwedge A \mid \varnothing \neq A \subset L(M), A \text{ finite}\}.$$

Then the set $H(M)$ of M-harmonic elements is defined as $H(M) = \overline{\hat{M}} \cap -\overline{\hat{M}}$ (**[8]**, **[9]**). Note that $-\hat{M} = \check{M}$ is the set of all finite suprema of elements of $L(M)$.

By $U(M)$ we denote the uniqueness closure of M, i.e. $e \in U(M)$ iff for all $\mu \in E'_+$, $\delta \in V(E)$ equality of μ and δ on M implies $\mu(e) = \delta(e)$ (**[10]**; cf. **[2]**).

A locally convex vector lattice E is called a locally convex M-space, if its topology is generated by a family $\{\|\ \|_\alpha\}$ of lattice seminorms which satisfy $\|e \vee f\|_\alpha = \|e\|_\alpha \vee \|f\|_\alpha$ for all positive $e, f \in E$ ("espaces de Kakutani" in **[5]**, cf. **[7**, II §7**]**). Such seminorms will be called M-seminorms in the sequel.

Received by the editors October 31, 1977 and, in revised form, April 3, 1978.
AMS (*MOS*) *subject classifications* (1970). Primary 46A40; Secondary 46E05.

THEOREM. *Let E be a locally convex M-space and $\varnothing \neq M$ a subset. Then* $H(M) = K(M) = K_\sigma(M) = K_0(M) = U(M)$.

Before proving the theorem let us compare its statement with results obtained by other authors.

When $E = C(X)$, X a compact metric space, and M is a point-separating subset containing a strict positive function, the equality $K_\sigma(M) = U(M)$ was proved by H. Berens and G. G. Lorenz in **[2]**. Since here M is an arbitrary subset of E, we have a new result even in this case.

The equality $H(M) = K(M)$ was proved by M. Wolff in **[9]** for locally convex vector lattices, if the closed linear hull $L(M)$ of M is nearly positively generated in the sense that $L(M) = \overline{L(M)_+ - L(M)_+}$. If in addition E is dual atomic (i.e. E' is atomic), then $K(M) = U(M)$ as was proved also by M. Wolff in **[10]**. Since locally convex M-spaces are not dual atomic in general, Wolff's theorem and our theorem cover different cases.

As a class of locally convex lattices where our theorem could be applied, one has the class of so-called weighted-function spaces $CV_0(X)$, which have been studied by many authors (**[3]**). They include well-known spaces as for example $E = C(X)$, X completely regular, equipped with the topology of compact convergence or $E = CB(X)$, the bounded continuous real functions on X, with the strict topology.

2. Proof of the Theorem. An essential tool in proving the Theorem is the concept of upper and lower envelopes of elements $e \in E$. This concept in connection with Korovkin-theorems is not new: it has been used already by H. Bauer (**[1]**), H. Berens and G. G. Lorenz (**[2]**) and by K. Donner (**[4]**).

Let E be a vector lattice equipped with a lattice seminorm $\| \ \|$ ($|e| \leqslant |f|$ implies $\|e\| \leqslant \|f\|$), E' its (topological) dual and B the positive part of the unit ball of E'. In the weak topology $\sigma(E', E)$ the set B is compact. The evaluation map $E \to A_0(B)$ sends elements $e \in E$ in continuous affine functions $\tilde{e}$ on B vanishing at $0 \in B$. Therefore we can define upper and lower envelopes of $e \in E$ as

$$\hat{e}(\mu) = \inf\{\mu(f) + r \mid f \in \hat{M}, r \in \mathbf{R}, \tilde{f} + r \geqslant \tilde{e}\},$$

$$\check{e}(\mu) = \sup\{\mu(f) + r \mid f \in \check{M}, r \in \mathbf{R}, \tilde{f} + r \leqslant \tilde{e}\}$$

for all $\mu \in B$ (pointwise order on B).[1]

We collect some simple properties of the envelopes in the following lemma:

LEMMA 1. *Let $e \in E$, $\mu \in B$.*

(i) $\check{e}(\mu) \leqslant \mu(e) \leqslant \hat{e}(\mu)$ *with equality on* $L(M)$.

(ii) $(-e)\hat{}(\mu) = -\check{e}(\mu)$.

(iii) $e \leqslant 0$ *implies* $\hat{e}(\mu) \leqslant 0$.

---

[1]To use this kind of envelope was suggested by K. Donner at the June 1977 meeting on "Riesz spaces and order bounded linear transformations" in Oberwolfach.

(iv) $\hat{e}(\mu) \leqslant \|e\|$.

(v) *The map* $e \mapsto \hat{e}(\mu)$ *is a sublinear functional on* E.

We omit the easy proof.

Now, let $V(E)_1 = V(E) \cap B$ and denote by $E(M)$ the set of all $e \in E$ the upper and lower envelopes of which coincide on $V(E)_1$.

LEMMA 2. *Let* $e \in E$, $\delta \in B$. *Then there exists a* $\mu \in B$ *such that* $\mu(e) = \hat{e}(\delta)$ and $\mu =_M \delta$.

PROOF. By (v) of Lemma 1 the mapping p_δ: $f \mapsto \hat{f}(\delta)$ is a sublinear functional on E. The linear functional μ_0 on $\mathbf{R} \cdot e$ defined by $\mu_0(re) = r\hat{e}(\delta)$ is dominated by p_δ; this is evident for $r \geqslant 0$. For $r < 0$ it follows by $\mu_0(-e) = -\hat{e}(\delta) \leqslant -\delta(e) = \delta(-e) \leqslant (-e)\hat{\ }(\delta) = p_\delta(-e)$ using (i) of Lemma 1.

The Hahn-Banach theorem yields an extension μ of μ_0 dominated by p_δ on E. By (iv) of Lemma 1, μ is continuous with norm $\leqslant 1$. By (iii) of Lemma 1 it is positive and thus belongs to B. Finally $\mu \leqslant p_\delta$ implies $\mu =_M \delta$ as $\delta(f) = \hat{f}(\delta)$ on $L(M)$.

LEMMA 3. *We have* $U(M) = E(M)$.

PROOF. Suppose e belongs to $U(M)$ and let $\delta \in V(E)_1$. By Lemma 2 there exists a $\mu \in B$ such that $\mu(e) = \hat{e}(\delta)$ and $\mu =_M \delta$. As $e \in U(M)$, $\mu(e) = \delta(e)$ and $\delta(e) = \hat{e}(\delta)$. Since also $-e \in U(M)$, $\delta(-e) = -\delta(e) = (-e)\hat{\ }(\delta) = -\check{e}(\delta)$ and $\delta(e) = \check{e}(\delta)$. Thus e belongs to $E(M)$.

Conversely, let $e \in E(M)$ and choose $\mu \in E'_+$, $\delta \in V(E)$ such that $\mu =_M \delta$. By multiplying by a positive constant, if necessary, we can assume $\mu \in B$, $\delta \in V(E)_1$. Now let $f \in \hat{M}$ and write $f = \bigvee_{i=1}^n f_i$ with $f_i \in L(M)$ $(i = 1, \dots, n)$. Since δ is a lattice homomorphism and since $\mu =_M \delta$, it follows that

$$\delta(f) = \bigvee_{i=1}^{n} \delta(f_i) = \bigvee_{i=1}^{n} \mu(f_i) \leqslant \mu(f)$$

and

$$\delta(f) + r \leqslant \mu(f) + r \quad \text{for all } r \in \mathbf{R}.$$

The definition of lower envelopes yields $\check{e}(\delta) \leqslant \check{e}(\mu)$. Similarly, one obtains $\hat{e}(\mu) \leqslant \hat{e}(\delta)$. Thus $e \in E(M)$ implies–using (i) of Lemma 1– $\mu(e) = \delta(e)$ and $e \in U(M)$.

LEMMA 4. *Let* E *have an* M-*seminorm. If* $e \in E$ *satisfies* $\delta(e) = \hat{e}(\delta)$ *for all* $\delta \in V(E)_1$, *then* $e \in \hat{M}$.

PROOF. First observe that $\tilde{e} \leqslant \tilde{f} + r$, $r \in \mathbf{R}$, implies $r \geqslant 0$ since $0 \in B$. Suppose $\varepsilon > 0$, $\delta \in V(E)_1$, by hypothesis on e there exist $f \in \hat{M}$, $0 \leqslant r \in \mathbf{R}$, such that $\tilde{e} \leqslant \tilde{f} + r$ and

$$\left(\tfrac{1}{2}\delta\right)(f) + r < \hat{e}\left(\tfrac{1}{2}\delta\right) + \tfrac{1}{2}\varepsilon = \tfrac{1}{2}(\delta(e) + \varepsilon) \leqslant \tfrac{1}{2}(\delta(f) + r + \varepsilon).$$

Thus $0 \leqslant r < \varepsilon$, $\tilde{e} \leqslant \tilde{f} + \varepsilon$ and $\delta(f) < \delta(e) + \varepsilon$. Hence the sets $U_f = \{\delta \in$

$V(E)_1|\delta(f) < \delta(e) + \varepsilon\}$ form a $\sigma(E', E)$-open covering of the $\sigma(E', E)$-compact set $V(E)_1$, when f varies in $\hat{M}$ such that $\tilde{e} \leqslant \tilde{f} + \varepsilon$. We therefore find finitely many $f_1, \ldots, f_n \in \hat{M}$ such that $\tilde{e} \leqslant \tilde{f}_i + \varepsilon$ for all $i = 1, \ldots, n$ and $V(E)_1 = \bigcup_{i=1}^{n} U_{f_i}$.

Let $f = \bigwedge_{i=1}^{n} f_i$; then $f \in \hat{M}$ and for an arbitrary $\delta \in V(E)_1$ we have

$$-\varepsilon + \delta(e) \leqslant \delta(f) = \min_{i=1,\ldots,n} \delta(f_i) < \delta(e) + \varepsilon.$$

Thus $\sup_{\delta \in V(E)_1}|\delta(f - e)| \leqslant \varepsilon$. Since E has an M-seminorm, $V(E)_1$ contains the extreme points of $\underline{B}$, so that $\sup_{\delta \in V(E)_1}|\delta(f - e)| = \|f - e\| \leqslant \varepsilon$. As $\varepsilon > 0$ was arbitrary, $e \in \hat{\bar{M}}$ as required.

Now we are able to prove the Theorem.

Proof of the Theorem. The inclusion $H(M) \subset K(M)$ was proved by M. Wolff in [8] for arbitrary locally convex vector lattices. The inclusions $K(M) \subset K_\sigma(M) \subset K_0(M) \subset U(M)$ follow immediately by the definitions of the respective spaces.

To prove $U(M) \subset H(M)$ we proceed as follows. Let $\{\| \ \|_\alpha\}_{\alpha \in A}$ be a saturated family of M-seminorms generating the topology of E and denote by E_α the space E seminormed by $\| \ \|_\alpha$. Furthermore let $U_\alpha(M)$, $E_\alpha(M)$ and $H_\alpha(M)$ denote the spaces $U(M)$, $E(M)$ and $H(M)$ constructed in E_α. By Lemma 3 we have $U_\alpha(M) = E_\alpha(M)$, by Lemma 4 and (ii) of Lemma 1, $E_\alpha(M) \subset H_\alpha(M)$. Now the assertion of the Theorem follows by

$$U(M) \subset \bigcap_{\alpha \in A} U_\alpha(M) = \bigcap_{\alpha \in A} E_\alpha(M) \subset \bigcap_{\alpha \in A} H_\alpha(M) = H(M).$$

In [10] M. Wolff proved $U(M) = K(M, I)$ for the identity Korovkin closure $K(M, I)$ of M in E (i.e. in the definition (*) only $F = E$ and $S = I$, the identity on E, is allowed), if E is a locally convex M-space. Thus our theorem together with Wolff's result implies:

Corollary. *If E is a locally convex M-space, then the identity Korovkin closure of M in E is universal and coincides both with the set of M-harmonic elements and the uniqueness closure of M.*

References

1. H. Bauer, *Theorems of Korovkin type for adapted spaces*, Ann. Inst. Fourier (Grenoble) **23** (1973), 245–260.

2. H. Berens and G. G. Lorentz, *Theorems of Korovkin type for positive linear operators on Banach lattices*, Approximation Theory (G. G. Lorentz, ed.), Academic Press, New York, 1973.

3. K.-D. Bierstedt, *Gewichtete Räume stetiger vektorwertiger Funktionen und das injektive Tensorprodukt*. I, J. Reine Angew. Math. **259** (1973), 186–210.

4. K. Donner, *Korovkin theorems for positive linear operators*, J. Approximation Theory **13** (1975), 443–450.

5. C. Portenier, *Espaces de Riesz, espaces de fonctions et espaces de sections*, Comment. Math. Helv. **46** (1971), 289–313.

6. H. H. Schaefer, *Topological vector spaces*, Springer-Verlag, Berlin and New York, 1974.

7. ______, *Banach lattices and positive operators*, Springer-Verlag, Berlin and New York, 1974.

8. M. Wolff, *Über die Korovkinhülle von Teilmangen in lokalkonvexen Vektorverbänden*, Math. Ann. **213** (1975), 97–108.

9. ______, *On the universal Korovkin closure of subsets in vector lattices*, J. Approximation Theory **22** (1978), 243–253.

10. ______, *On the theory of approximation by positive operators in vector lattices*, Functional Analysis: Surveys and Recent Results (K.-D. Bierstedt, B. Fuchssteiner, eds.), North-Holland Mathematics Studies, vol. 27, Amsterdam, 1977.

FACHBEREICH MATHEMATIK, TECHNISCHE HOCHSCHULE, D-6100 DARMSTADT, FEDERAL REPUBLIC OF GERMANY