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LOCAL PROPERTIES OF QUOTIENT ANALYTIC SPACES

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ABSTRACT. Let $T := \mathbf{C}/\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ be a complex 1-torus and E_n the set of all elliptic functions of order n. Then M. Namba showed that E_n is a $2n$-dimensional complex manifold. Let Aut T be the automorphism group of T, then Aut T is a 1-dimensional compact complex Lie group and the orbit space $E_n/\mathrm{Aut}\, T$ is an analytic space. In this paper, we shall show that $E_n/\mathrm{Aut}\, T$ has only rational singularities and if $n \geqslant 5$, $E_n/\mathrm{Aut}\, T$ is rigid.

1. Introduction. Let M be a complex manifold and G a properly discontinuous transformation group on M. Then H. Cartan [2] showed that the quotient space M/G is a normal analytic space and D. Burns [1] has proved that M/G is rational. Moreover M. Schlessinger [7], [8] showed that if codim $F(G) \geqslant 3$, M/G has only rigid singularities, where

$$F(G) := \bigcup_{g \in G - \{e\}} F(g), F(g) := \{x \in M; gx = x\}$$

is the set of all fixed points.

In this paper, let M be also a complex manifold and G a complex Lie transformation group whose action is proper on M. Then H. Holmann [4] proved that the quotient space M/G is a normal analytic space. We shall show the following

THEOREM 1. *M/G has only rational singularities. Moreover if* codim $S(M/G) \geqslant 3$, *M/G is rigid, where $S(M/G)$ is the set of all singular points of M/G.*

Let $\omega_1, \omega_2 \in \mathbf{C}$; $\mathrm{Im}(\omega_2/\omega_1) > 0$, $T := \mathbf{C}/\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ be a complex 1-torus and E_n the set of all elliptic functions of order n. Then E_n is a $2n$-dimensional complex manifold [5] and the automorphism group Aut T acts on E_n naturally. Then we have

THEOREM 2. *$E_n/\mathrm{Aut}\, T$ is rational and if $n \geqslant 5$, $E_n/\mathrm{Aut}\, T$ has only rigid singularities.*

2. The proof of Theorem 1. We shall give some definitions. Let $(X, \mathcal{O}_X)$ be an analytic space and π: $(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to (X, \mathcal{O}_X)$ a resolution of singularities of X. Then $x \in X$ is called a rational singularity if $(R^i\pi_*\mathcal{O}_{\tilde{X}})_x = 0$ for any $i > 0$

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(cf. [1]). $x \in X$ is called a rigid singularity if any local flat deformation of (X, x) is locally trivial (cf. [7], [8]).

Let R be an equivalence relation on X. We identify the graph of R with R. Then R is called open (resp. proper, finite) if the natural projection p_1: $R \to X$ is open (resp. proper, finite). And R is called analytic if the graph R is an analytic set in $X \times X$ (cf. [4], [9]).

Let M be a complex manifold and G a complex Lie transformation group on M. Then G is called proper on M if the graph mapping Ψ: $G \times M \to M \times M$, $\Psi(g, x) := (gx, x)$ is proper. (This is different from the above definition that an equivalence relation R is proper.) Let $G(x) := \{gx; g \in G\}$ be the orbit through x and $G_x := \{g \in G; gx = x\}$ an isotropy group at x.

We shall prove Theorem 1 and its proof is essentially owing to the technique by H. Holmann [3], [4]. Let Φ: $G \times M \to M$, $\Phi(g, x) := gx$ be the natural holomorphic mapping. Then $G(x) = \Phi(G \times \{x\})$ is a complex submanifold of M and biholomorphic to G/G_x. Since G is proper on M, M/G is Hausdorff with respect to the quotient topology. Hence from Hilfssatz 7 of [4], for any $x \in M$ there exist an open connected neighborhood $U \subset M$ of x and a submanifold $N \subset U$ such that

(1) $x \in N$,

(2) $U \cong (G(x) \cap U) \times N$,

(3) $G(y) \cap U \cong (G(x) \cap U) \times N_y$ for any $y \in N$, where $N_y := G(y) \cap N$ is finite and $N_x = \{x\}$. Let R be an equivalence relation on N defined by $R(y) := N_y$ for any $y \in N$. Then R is open proper finite analytic (if we choose U small enough) and M/G is locally isomorphic to N/R (cf. the proof of Satz 15 in [4]).

For any $g \in G_x$, there exists an open connected neighborhood $V \subset N$ of x such that $g' := qg$: $V \to g'(V) \subset N$ is biholomorphic, where q: $U \to N$ is the natural projection and the inverse of g' is

$$(g')^{-1} := qg^{-1}.$$

Let $G'(V)$ be the set of all biholomorphic mappings on V induced from G_x for any connected neighborhood $V \subset N$ of x. Since G is proper on M, there exists an open connected neighborhood $V \subset N$ of x such that $G'(V)$ is a finite transformation group on V and R coincides with the equivalence relation induced by $G'(V)$ on V (cf. the proof of Satz 19 in [4]). Therefore M/G is locally isomorphic to $V/G'(V)$ and so M/G has only rational singularities. If we show the following lemma, the proof of Theorem 1 is complete.

LEMMA 1. *Let* Γ *be a finite subgroup of* $\mathrm{GL}(n, \mathbf{C})$. *If* $\operatorname{codim} S(\mathbf{C}^n/\Gamma) \geqslant 3$, $\mathbf{C}^n/\Gamma$ *is rigid.*

PROOF. Let Γ_0 be the normal subgroup of Γ generated by reflections, where $g \in \mathrm{GL}(n, \mathbf{C})$ is called a reflection if g is order-finite and $\operatorname{codim} F(g) = 1$.

Then $\mathbf{C}^n/\Gamma_0$ is biholomorphic to $\mathbf{C}^n$ and Γ/Γ_0 acts on $\mathbf{C}^n/\Gamma_0$ naturally. Moreover $\mathbf{C}^n/\Gamma \cong (\mathbf{C}^n/\Gamma_0)/(\Gamma/\Gamma_0)$ and Γ/Γ_0 has no reflection (cf. [6]). Thus we may assume that Γ has no reflection. Then $p(F(\Gamma)) = S(\mathbf{C}^n/\Gamma)$, where p: $\mathbf{C}^n \to \mathbf{C}^n/\Gamma$ is the projection.

In fact, $p(F(\Gamma)) \supset S(\mathbf{C}^n/\Gamma)$ is clear. Let $x \in \mathbf{C}^n$ and $p(x) \notin S(\mathbf{C}^n/\Gamma)$, then there exists a nonsingular neighborhood $U \subset \mathbf{C}^n/\Gamma$ of $p(x)$ such that p: $p^{-1}(U) \to U$ is a finite covering between complex manifolds. We have

$$F(\Gamma) \cap p^{-1}(U) = \{t \in p^{-1}(U); Jp(t) = 0\},$$

where Jp is the Jacobian of p. Since codim $F(\Gamma) \geqslant 2$, $F(\Gamma) \cap p^{-1}(U)$ must be empty. Thus $x \notin F(\Gamma)$. Therefore codim $F(\Gamma) =$ codim $S(\mathbf{C}^n/\Gamma) \geqslant 3$ and so $\mathbf{C}^n/\Gamma$ is rigid.

REMARK. If G is not proper on M, locally M/G is not always isomorphic to $V/G'(V)$.

3. The proof of Theorem 2. Since Aut T is compact, Aut T is proper on E_n. Hence the first part of Theorem 2 is straightforward from Theorem 1. Thus we must calculate codim $S(E_n/\mathrm{Aut}\ T)$.

Let S_n be the n-symmetric group. Then $S_n \times S_n$ acts on $\mathbf{C}^{2n+1}$ as follows:

$$(S_n \times S_n) \times \mathbf{C}^{2n+1} \to \mathbf{C}^{2n+1},$$
$$(g_1, g_2, \alpha, \beta, \gamma) \rightsquigarrow (g_1\alpha, g_2\beta, \gamma),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$. We define

$$\phi\colon \mathbf{C}^{2n+1} \to \mathbf{C}^{2n+1},$$
$$\phi(\alpha, \beta, \gamma) := (\sigma_1(\alpha), \ldots, \sigma_n(\alpha), \sigma_1(\beta), \ldots, \sigma_n(\beta), \gamma),$$

where σ_i is the i-fundamental symmetric function ($i = 1, \ldots, n$). Then we have $\mathbf{C}^{2n+1}/(S_n \times S_n) \cong \phi(\mathbf{C}^{2n+1}) = \mathbf{C}^{2n+1}$.

For $\omega_1, \omega_2 \in \mathbf{C}$; $\mathrm{Im}(\omega_2/\omega_1) > 0$, we put $\Delta := \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$, $\Delta^* := \Delta - \{0\}$ and

$$\tilde{S}_\Omega := \Big\{(\alpha, \beta, \gamma) \in \mathbf{C}^{2n+1};\ \sum_{i=1}^n \alpha_i - \sum_{i=1}^n \beta_i = \Omega,\ \gamma \neq 0,\ \alpha_i - \beta_j \notin \Delta,$$
$$\alpha_i - \alpha_j \notin \Delta^*,\ \beta_i - \beta_j \notin \Delta^*\ (i, j = 1, \ldots, n)\Big\}$$

for $\Omega \in \Delta$. Since $\tilde{S}_\Omega$ is an $S_n \times S_n$-invariant complex manifold,

$$S_\Omega := \phi(\tilde{S}_\Omega) = \{(x, y, z) \in \mathbf{C}^{2n+1};\ x_1 - y_1 = \Omega,\ z \neq 0,\ \alpha_i - \beta_j \notin \Delta,$$
$$\alpha_i - \alpha_j \notin \Delta^*,\ \beta_i - \beta_j \notin \Delta^*\ (i, j = 1, \ldots, n)\}$$

is also a $2n$-dimensional complex manifold, where α_i, β_i are roots of

$$X^n - x_1X^{n-1} + x_2X^{n-2} + \cdots + (-1)^n x_n = 0, \tag{1}$$
$$X^n - y_1X^{n-1} + y_2X^{n-2} + \cdots + (-1)^n y_n = 0 \tag{2}$$

respectively. And $\tilde{S} := \bigcup_{\Omega \in \Delta} \tilde{S}_\Omega$, $S := \bigcup_{\Omega \in \Delta} S_\Omega$ are complex manifolds.

By the classical theory of elliptic functions, for any $f(w) \in E_n$, let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ be zeroes and poles of $f(w)$ respectively, then

$$\sum_{i=1}^{n} \alpha_i - \sum_{i=1}^{n} \beta_i = \Omega \in \Delta, \qquad \Omega = p_0\omega_1 + q_0\omega_2,$$

$$f(w) = \gamma \exp 2(p_0\eta_1 + q_0\eta_2)w \prod_{i=1}^{n} \frac{\sigma(w - \alpha_i)}{\sigma(w - \beta_i)},$$

where $\eta_j := \zeta(\frac{1}{2}\omega_j)(j = 1, 2)$ and σ, ζ are the Weierstrass functions. We define $\tilde{F}$: $\tilde{S} \to E_n$

$$\tilde{F}(\alpha, \beta, \gamma) := \gamma \exp 2(p_0\eta_1 + q_0\eta_2)w \prod_{i=1}^{n} \frac{\sigma(w - \alpha_i)}{\sigma(w - \beta_i)},$$

where

$$\sum_{i=1}^{n} \alpha_i - \sum_{i=1}^{n} \beta_i = p_0\omega_1 + q_0\omega_2 \in \Delta.$$

Since $\tilde{F}$ is $S_n \times S_n$-invariant, $\tilde{F}$: $\tilde{S} \to E_n$ induces F: $S \to E_n$

$$F(x, y, z) := z \exp 2(p_0\eta_1 + q_0\eta_2)w \prod_{i=1}^{n} \frac{\sigma(w - \alpha_i)}{\sigma(w - \beta_i)},$$

where $x_1 - y_1 = p_0\omega_1 + q_0\omega_2 \in \Delta$ and α_i, β_i are roots of (1), (2) respectively. Then we can show the following lemma (cf. [5, Remark, p. 75]).

LEMMA 2. *F: $S \to E_n$ is surjective open holomorphic and locally biholomorphic.*

We shall calculate codim $S(E_n/\mathrm{Aut}\, T)$ using this locally biholomorphic mapping F. Here we assume that $\omega_2/\omega_1 \not\cong e^{\pi i/2}$, $e^{2\pi i/3}$ under the modular group. Then Aut T has two connected components T_0, T_1 and T_0 is isomorphic to T as Lie group. We put

$$g_1(x_1, x_2, x_3, n) := -x_3 + \frac{n-2}{n} x_1x_2 - \frac{(n-1)(n-2)}{3n^2} x_1^3,$$

$$g_2(x_1, x_2, x_3, x_4, x_5, n) := -x_5 + \frac{n-4}{n} x_1x_4 - \frac{(n-3)(n-4)}{n^2} x_1^2x_3$$
$$+ \frac{2(n-2)(n-3)(n-4)}{3n^3} x_1^3x_2 - \frac{(n-1)(n-2)(n-3)(n-4)}{5n^4} x_1^5,$$

$$\rho(x, \lambda) := \lambda^n + x_1\lambda^{n-1} + \cdots + x_{n-1}\lambda + x_n, \; x = (x_1, \ldots, x_n),$$

$$a_i(x, \lambda) := \frac{1}{(n-i)!} \rho^{(n-i)}(x, \lambda)$$
$$= \binom{n}{i}\lambda^i + \binom{n-1}{i-1}x_1\lambda^{i-1} + \cdots + (n - (i-1))x_{i-1}\lambda + x_i,$$

$$a_i'(x, \lambda) := (-1)^i a_i(x, \lambda), \qquad t := -(\lambda + 2x_1/n).$$

Then we have the following lemma by long elementary calculation.

LEMMA 3.

$$a_1'(x, \lambda) = a_1(x, t), a_2'(x, \lambda) = a_2(x, t),$$
$$a_3'(x, \lambda) = a_3(x, t) + 2g_1(x_1, x_2, x_3, n),$$
$$a_4'(x, \lambda) = a_4(x, t) + 2(n-3)(t + x_1/n)g_1(x_1, x_2, x_3, n),$$
$$a_5'(x, \lambda) = a_5(x, t) + (n-3)(n-4)(t^2 + 2x_1t/n)g_1(x_1, x_2, x_3, n) + 2g_2(x_1, x_2, x_3, x_4, x_5, n),$$
$$a_6'(x, \lambda) = a_6(x, t) + (n-3)(n-4)\left(\frac{n-5}{3}t^3 + \frac{n-5}{n}x_1t^2 - \frac{1}{3n^2}x_1^2\right)g_1 \cdot (x_1, x_2, x_3, n) + 2(n-5)(t + x_1/n)g_2(x_1, x_2, x_3, x_4, x_5, n).$$

In general,

$$a_i'(x, \lambda) = a_i(x, t) + \sum_{j=0}^{i-3}\left(\sum_{k=0}^{i-j}(-1)^k\binom{n-k}{i-k}\binom{i-k}{i-j-k}\left(\frac{2}{n}x_1\right)^{i-j-k} \cdot x_k - \binom{n-i+j}{j}x_{i-j}\right)t^j$$
$$(i = 1, \ldots, n).$$

Let

$$A_n := \{(x, y, z) \in S_0;\ a_i'(x, -2x_1/n) = x_i,\ a_i'(y, -2y_1/n) = y_i\ (i = 1, \ldots, n)\}.$$

(*If* $i = 1, 2$, $a_i'(x, -2x_1/n) = x_i$ *and* $a_i'(y, -2y_1/n) = y_i$ *are always true.*) *Then* $A_n = \{(x, y, z) \in S_0;\ g_1(x_1, x_2, x_3, n) = g_1(y_1, y_2, y_3, n) = 0\}$ *for* $n = 3, 4$ *and* $A_n = \{(x, y, z) \in S_0;\ g_1(x_1, x_2, x_3, n) = g_2(x_1, x_2, x_3, x_4, x_5, n) = g_1(y_1, y_2, y_3, n) = g_2(y_1, y_2, y_3, y_4, y_5, n) = 0\}$ *for* $n = 5, 6$. *Thus if* $n \geqslant 5$, *we have* codim $A_n \geqslant 4$.

Now we shall prove the rest of Theorem 2. For any $f(w) \in E_n$, there exists $(\alpha, \beta, \gamma) \in \tilde{S}_\Omega$ such that

$$f(w) = \gamma \exp 2(p_0\eta_1 + q_0\eta_2)w \prod_{i=1}^{n} \frac{\sigma(w - \alpha_i)}{\sigma(w - \beta_i)}, \qquad \Omega = p_0\omega_1 + q_0\omega_2.$$

Let $B_n := \{(x, y, z) \in S;$ (1) and (2) have no simple root$\}$, then B_n is an analytic set in S. If $(a, b, \gamma) := \phi(\alpha, \beta, \gamma) \in (S - B_n) \cup S_0$, we may assume that $(a, b, \gamma) \in S_0$. Then

$$f(w) = \gamma \prod_{i=1}^{n} \frac{\sigma(w - \alpha_i)}{\sigma(w - \beta_i)}, \qquad a_1 = b_1.$$

For any $a(w) \in T_0$; $a(w) = w - \lambda$, we have

$$f \circ a(w) = \gamma \prod_{i=1}^{n} \frac{\sigma(w - \lambda - \alpha_i)}{\sigma(w - \lambda - \beta_i)}.$$

Hence $(\lambda + \alpha, \lambda + \beta, \gamma) \in \tilde{S}_0 \cap \tilde{F}^{-1}(f \circ a(w))$ and $\phi(\lambda + \alpha, \lambda + \beta, \gamma) = (a_1(a, \lambda), \ldots, a_n(a, \lambda), a_1(b, \lambda), \ldots, a_n(b, \lambda), \gamma) \in S_0 \cap F^{-1}(f \circ a(w))$. For any $a(w) \in T_1$; $a(w) = -w - \lambda$, we have

$$f \circ a(w) = \gamma \prod_{i=1}^{n} \frac{\sigma(w + \lambda + \alpha_i)}{\sigma(w + \lambda + \beta_i)}.$$

Hence $(-\lambda - \alpha, -\lambda - \beta, \gamma) \in \tilde{S}_0 \cap \tilde{F}^{-1}(f \circ a(w))$ and $\phi(-\lambda - \alpha, -\lambda - \beta, \gamma) = (a_1'(a, \lambda), \ldots, a_n'(a, \lambda), a_1'(b, \lambda), \ldots, a_n'(b, \lambda), \gamma) \in S_0 \cap F^{-1}(f \circ a(w))$. Namely the orbit $(\mathrm{Aut}\, T)(f(w))$ is locally described by $\{(a_1(a, \lambda), \ldots, a_n(a, \lambda), a_1(b, \lambda), \ldots, a_n(b, \lambda), \gamma);\ \lambda \in T\} \cup \{(a_1'(a, \lambda), \ldots, a_n'(a, \lambda), a_1'(b, \lambda), \ldots, a_n'(b, \lambda), \gamma);\ \lambda \in T\}$ in S_0. Hence using Lemma 3, we can show that $(\mathrm{Aut}\, T)_{f(w)} = \{w\}$ if $(a, b, \gamma) \notin A_n$. Then $E_n/\mathrm{Aut}\, T$ is nonsingular at $p(f(w))$ by the proof of Theorem 1. If $n \geqslant 5$, we have codim $A_n \geqslant 4$ and codim $B_n \geqslant n \geqslant 5$. Therefore codim $S(E_n/\mathrm{Aut}\, T) \geqslant 3$ and so $E_n/\mathrm{Aut}\, T$ is rigid.

In the case $\omega_2/\omega_1 \cong e^{\pi i/2}$ or $e^{2\pi i/3}$, we can also show that if $n \geqslant 5$, codim $S(E_n/\mathrm{Aut}\, T) \geqslant 3$ by almost the same argument as above.

EXAMPLE. Let (w) be the Weierstrass pe-function. Then

$$\wp(w)^2 = \frac{1}{\sigma(\alpha)^4} \frac{\sigma(w - \alpha)^2 \sigma(w + \alpha)^2}{\sigma(w)^4} \in E_4\ (\wp(\alpha) = 0)$$

and $(\mathrm{Aut}\, T)_{\wp(w)^2} = \{w, -w\}$. Thus $(\alpha, \alpha, -\alpha, -\alpha, 0, 0, 0, 0, 1/\sigma(\alpha)^4) \in \tilde{S}_0 \cap \tilde{F}^{-1}(\wp(w)^2)$ and $(0, -2\alpha^2, 0, \alpha^4, 0, 0, 0, 0, 1/\sigma(\alpha)^4) \in S_0 \cap F^{-1}(\wp(w)^2)$. By Lemma 3 and the proof of Theorems 1 and 2, we may take $N := \{x_1 = y_1 = 0\} \subset S_0$ and for any $(x, y, z) = (0, x_2, x_3, x_4, 0, y_2, y_3, y_4, z) \in N$, the orbit $(\mathrm{Aut}\, T)(F(x, y, z))$ is locally described in S_0 as follows:

$$\begin{aligned}&\{(4\lambda, 6\lambda^2 + x_2, 4\lambda^3 + 2x_2\lambda + x_3, \lambda^4 + x_2\lambda^2 + x_3\lambda + x_4, 4\lambda, 6\lambda^2 + y_2,\\ &\qquad 4\lambda^3 + 2y_2\lambda + y_3, \lambda^4 + y_2\lambda^2 + y_3\lambda + y_4, z);\ \lambda \in T\}\\ &\cup \{(-4\lambda, 6\lambda^2 + x_2, -4\lambda^3 - 2x_2\lambda - x_3, \lambda^4 + x_2\lambda^2 + x_3\lambda + x_4, -4\lambda,\\ &\qquad 6\lambda^2 + y_2, -4\lambda^3 - 2y_2\lambda - y_3, \lambda^4 + y_2\lambda^2 + y_3\lambda + y_4, z);\ \lambda \in T\}.\end{aligned}$$

Hence $N_{(x,y,z)} = \{(0, x_2, x_3, x_4, 0, y_2, y_3, y_4, z), (0, x_2, -x_3, x_4, 0, y_2, -y_3, y_4, z)\}$. Therefore $E_4/\mathrm{Aut}\, T$ is isomorphic to $\{w^2 - uv = 0\} \times \mathbf{C}^5 \subset \mathbf{C}^8$ at $p(\wp(w)^2)$ and so not rigid.

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