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ON THE LOCAL SPECTRA OF SEMINORMAL OPERATORS[1]

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ABSTRACT. Two theorems on the local spectra of seminormal operators are deduced. In the first theorem it is shown that when T is hyponormal any solution of the equation $(T - \lambda)x(\lambda) = x$ on an open set in the plane is necessarily analytic. The second theorem establishes the existence of vectors with small local spectra for a cohyponormal operator with a finite rank self-commutator.

Let T be an operator on a Hilbert space $\mathcal{H}$. For x a vector in $\mathcal{H}$ the local resolvent $(T - \lambda)^{-1}x$ is analytic on the resolvent set $\rho(T)$. In case any two analytic extensions of $(T - \lambda)^{-1}x$ agree on their common domain, then the operator T is said to have the single valued extension property. When T has the single valued extension property the domain of the maximal analytic continuation $\tilde{x}$ of the local resolvent is denoted by $\rho_T(x)$. The complementary set $\sigma_T(x) = \mathbf{C} \setminus \rho_T(x)$ is called the local spectrum of x. If T has the single valued extension property, then for δ a closed subset in the plane $X_T(\delta)$ will denote the linear manifold consisting of vectors x such that $\sigma_T(x)$ is contained in δ.

An operator S on $\mathcal{H}$ is called seminormal in case its selfadjoint self-commutator $D = S^*S - SS^*$ is semidefinite. In case $D \geqslant 0$, the operator S is called hyponormal and when $D \leqslant 0$, the operator S is called cohyponormal. There is a marked difference between the local spectral theories for hyponormal and cohyponormal operators. It is known that if H is hyponormal, then H has the single valued extension property. Further $X_H(\delta)$ is closed for all closed $\delta \subset \mathbf{C}$. This latter result appears in Stampfli [7] with the hypothesis that the spectrum $\sigma(H)$ consists of continuous spectrum. Radjabalipour [6] has shown that the latter hypothesis on the spectrum of H is unnecessary. It is apparently unknown (excepting trivial cases where H^* has eigenvalues) whether δ can be chosen so that $X_H(\delta)$ provides a nontrivial invariant subspace for the operator H. On the other hand when C is a nonnormal cohyponormal operator with the single valued extension property there are nonzero vectors with local spectra a proper subset of $\sigma(C)$. Whether $X_C(\delta)$ is closed for closed $\delta \subset \mathbf{C}$ is not known. These vectors with small local spectra are provided by a result of Putnam [5] which asserts that any vector in the range of $CC^* - C^*C$ belongs to the range of $C - \lambda$, for all $\lambda \in \mathbf{C}$. As we shall see below there are no nonzero vectors in the range of $H - \lambda$, for all

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$\lambda \in \mathbf{C}$, when H is a hyponormal operator.

Let T be an operator on $\mathcal{H}$ and $\delta \subset \mathbf{C}$ closed. The notation

$$Z_T(\delta) = \bigcap_{\lambda \notin \delta} (T - \lambda)\mathcal{H}$$

will be employed.

THEOREM 1. *Let H be a hyponormal operator on $\mathcal{H}$ and $\delta \subset \mathbf{C}$ closed. Then $Z_H(\delta) = X_H(\delta)$.*

The above result answers a question of Radjabalipour [6]. Theorem 1 improves a result of Stampfli and Wadhwa [8]. The possibility that H be a normal operator is not ruled out. For the case where H is normal the result in Theorem 1 appears in Putnam [6] and implicitly in [2].

An immediate corollary is the following.

COROLLARY. *Let H be hyponormal. Then $Z_H(\phi) = (0)$.*

Although in the nonnormal cohyponormal case it is possible to find vectors with local spectra a proper subset of the spectrum, it is not easy to control the location of the local spectra. In this direction we establish:

THEOREM 2. *Let C be a cohyponormal operator on $\mathcal{H}$ possessing the single valued extension property and having a finite rank self-commutator. Let Δ be an open disc such that $\Delta \cap \sigma(C) \neq \varnothing$. There exists a vector x in $\mathcal{H}$ such that $\sigma_C(x) \subset \Delta$.*

1. Preliminaries to Theorem 1. In this section we record lemmas needed for the proof of Theorem 1.

LEMMA 1. *Let T be an operator without point spectra and $x \in \mathcal{H}$. Let Δ_x be the set of λ in $\mathbf{C}$ for which there is a vector $x(\lambda)$ satisfying $(T - \lambda)x(\lambda) = x$. Then $\|x(\lambda)\|$ is a lower semicontinuous function on Δ_x.*

PROOF. Write $T - \lambda$ as T_λ. Let $T_\lambda^* = U_\lambda P_\lambda$ be the polar factorization of T_λ^*. The operator U_λ is coisometric with null space the same as T_λ^* and $P_\lambda = (T_\lambda T_\lambda^*)^{1/2}$. Let $P_\lambda = \int t\, dE^\lambda(t)$ denote the spectral resolution of P_λ. It is easy to verify that if $(T - \lambda)x(\lambda) = x$, then

$$\|x(\lambda)\|^2 = \int t^{-2} d\|E^\lambda(t)x\|^2.$$

This last identity and in fact the lower semicontinuity of $\|x(\lambda)\|$ follow from the equalities

$$\int t^{-2} d\|E^\lambda(t)x\|^2 = \lim_{\mu \to 0^+} \int (t + \mu)^{-2} d\|E^\lambda(t)x\|^2 = \lim_{\mu \to 0^+} \left((P_\lambda + \mu)^{-2}x, x\right).$$

The lower semicontinuity is implied by the continuity of $((P_\lambda + \mu)^{-2}x, x)$ as a function of λ, when $\mu > 0$ is fixed. This ends the proof.

The following lemma appears in Stampfli [7, Theorem 1]. Actually as stated

in Stampfli one requires the spectrum of T to consist of continuous spectrum. This hypothesis is removed in Radjabalipour [6].

LEMMA 2. *Let H be hyponormal without eigenvalues and let $x \in \mathcal{H}$. Then*

$$\|\tilde{x}(\lambda)\| \leqslant \frac{\|x\|}{\operatorname{dist}(\lambda, \sigma_H(x))}.$$

The final lemma which we need is stated in Radjabalipour [6, Theorem 1] (see, also Stampfli and Wadhwa [8]).

LEMMA 3. *Let H be a hyponormal operator without eigenvalues and $\delta \subset \mathbf{C}$ be closed. Suppose there is a bounded function $x(\lambda)$ satisfying $(H - \lambda)x(\lambda) = x$, $\lambda \notin \delta$. Then $x(\lambda)$ is analytic on $\mathbf{C} \setminus \delta$.*

2. Proof of Theorem 1. In the proof of the theorem it can be assumed that the operator H has no eigenvalues. Let Δ be a closed disc. Assume that $x \in H$ satisfies $x \in \bigcap_{\lambda \in \Delta}(H - \lambda)\mathcal{H}$. For $\lambda \in \Delta$, the vector $x(\lambda)$ will be assumed to satisfy $(H - \lambda)x(\lambda) = x$. It will be shown that the interior Δ° of Δ is contained in $\rho_H(x)$. The theorem follows easily from this last statement.

Suppose to the contrary that $\Delta^\circ \cap \sigma_H(x) \neq \varnothing$.

For $n = 1, 2, \ldots,$ set $F_n = \{\lambda \in \Delta \cap \sigma_H(x)\colon \|x(\lambda)\| \leqslant n\}$. It follows from the lower semicontinuity of $\|x(\lambda)\|$ (Lemma 1) that F_n is closed and by hypothesis $\bigcup_{n=1}^{\infty} F_n = \Delta \cap \sigma_H(x)$. The Baire Category Theorem implies that for some m the set $F_m \cap \Delta^\circ$ has interior in the relative topology on $\Delta \cap \sigma_H(x)$. This means there is an open disc $D \subset \Delta^\circ$ with center in $\sigma_H(x)$ so that $\|x(\lambda)\| \leqslant m$, for all $\lambda \in D \cap \sigma_H(x)$. Let D' be the disc with same center as D and radius equal to one-half the radius of D. Let $\lambda_0 \in D' \cap \rho_H(x)$ (if such a λ_0 exists) and choose γ_0 in $D \cap \sigma_H(x)$ such that $|\lambda_0 - \gamma_0| = \operatorname{dist}(\lambda_0, \sigma_H(x))$. Set $z_0 = x(\gamma_0) = (H - \gamma_0)^{-1}x$. It is easy to see that $\sigma_H(x) = \sigma_H(z_0)$ and for $\lambda \in \rho_H(x)$

$$z(\lambda) = \frac{x(\lambda) - x(\gamma_0)}{\lambda - \gamma_0}$$

satisfies $(H - \lambda)z(\lambda) = z_0$. It follows from Lemma 2 that

$$\|z(\lambda_0)\| = \frac{\|x(\lambda_0) - z_0\|}{|\lambda_0 - \gamma_0|} \leqslant \frac{\|z_0\|}{\operatorname{dist}(\lambda_0, \sigma_H(z_0))}.$$

Since $\operatorname{dist}(\lambda_0, \sigma_H(z_0)) = \operatorname{dist}(\lambda_0, \sigma_H(x)) = |\lambda_0 - \gamma_0|$, we have $\|x(\lambda_0)\| \leqslant 2\|z_0\| \leqslant 2m$. In any case $\|x(\lambda)\| \leqslant 2m$, for all $\lambda \in D'$, and from Lemma 3 we conclude that $x(\lambda)$ is analytic on D'. This contradicts the assumption that the center of D' belongs to $\sigma_H(x)$ and ends the proof.

3. Proof of Theorem 2. Let C be a cohyponormal operator with the single valued extension property. It will be assumed that the self-commutator of C has finite rank N and we will write $D = CC^* - C^*C = \sum_{i=1}^{N}(\ ,\psi_i)\psi_i$ where ψ_i

is orthogonal to ψ_j for $i \neq j$. Let $\pi_0(C)$ denote the collection of eigenvalues of C. Putnam [5] has shown that there are weakly continuous functions $\tilde{\psi}_i$: $\mathbf{C} \setminus \pi_0(C) \to \mathcal{H}$ such that $(C - z)\tilde{\psi}_i(z) = \psi_i$, $\|\tilde{\psi}_i(z)\| \leqslant 1$, $i = 1, \ldots, N$, $z \in \mathbf{C} \setminus \pi_0(C)$. Below it will be shown that if Δ is a disc with $\Delta \cap [\sigma(C) \setminus \pi_0(C)] \neq \varnothing$, then one of the functions $\tilde{\psi}_1, \ldots, \tilde{\psi}_N$ fails to be analytic on Δ.

In the remainder of this section it will be assumed that the operator C is completely nonnormal. This means that there are no nontrivial reducing subspaces of $\mathcal{H}$ on which C is a normal operator. This ensures (among other things) that $\pi_0(C^*) = \varnothing$. Let $z \in \sigma(C) \setminus \pi_0(C)$ and let $C - z = W_z P_z^{1/2}$ be the polar factorization of $C - z$; here, $P_z = (C - z)^*(C - z)$ and W_z is unitary. It is easy to verify that $W_z P_z = (P_z + D) W_z$.

A result of Krein [3] asserts that existence of a measurable function δ_z with compact support in $[0, \infty)$ satisfying $0 \leqslant \delta_z \leqslant N$ such that

$$\det\left[I_N - \left[\left((P_z + D - \lambda)^{-1}\psi_i, \psi_j\right)\right]_{N \times N}\right] = \exp\left\{-\int_0^\infty \frac{\delta_z(t)}{t - \lambda}\, dt\right\}, \quad (1)$$

for all λ such that $\operatorname{Re} \lambda \notin [0, \infty)$. In the preceding equation I_N denotes the $N \times N$ identity matrix and $[((P_z + D - \lambda)^{-1}\psi_i, \psi_j)]_{N \times N}$ is the $N \times N$ matrix with ij entry $((P_z + D - \lambda)^{-1}\psi_i, \psi_j)$.

We would like to take the limit in equation (1) as $\lambda \to 0^-$. Our argument proceeds as in Putnam [5]. One notes for $k = 1, \ldots, N$ and $\lambda < 0$

$$\begin{aligned}\left\|(P_z + D - \lambda)^{-1/2}\psi_k\right\|^2 &= \left((P_z + D - \lambda)^{-1}\psi_k, \psi_k\right) \\ &\leqslant \left((D - \lambda)^{-1}\psi_k, \psi_k\right) = \frac{\|\psi_k\|^2}{\|\psi_k\|^2 - \lambda} \leqslant 1.\end{aligned}$$

It follows that ψ_k belongs to the domain of $(P_z + D)^{-1/2}$ and

$$\lim_{\lambda \to 0^-} (P_z + D - \lambda)^{-1/2}\psi_k = (P_z + D)^{-1/2}\psi_k, \qquad k = 1, \ldots, N.$$

Taking the limit as $\lambda \to 0^-$ in equation (1) one obtains

$$\det\left[I_N - \left[\left((P_z + D)^{-1/2}\psi_i, (P_z + D)^{-1/2}\psi_j\right)\right]_{N \times N}\right] = \exp - \int_0^\infty \frac{\delta_z(t)}{t}\, dt.$$

Using the facts that $W_z W_z^* = I$ and $W_z^*(P_z + D)^{-1/2}\psi_k = \tilde{\psi}_k(z)$ we can write

$$\det\left[I_N - \left[\left(\tilde{\psi}_i(z), \tilde{\psi}_j(z)\right)\right]_{N \times N}\right] = \exp\left[-\int_0^\infty \frac{\delta_z(t)}{t}\, dt\right]. \quad (2)$$

Carey and Pincus [1] have identified the function δ_z in the following manner. There exists an integrable function G_z^P on the cylinder $[0, \infty) \times T$ such that

$$\det\left[I_N + \left(W_z(P_z + D - \lambda)^{-1}(W_z - \tau)^{-1}\psi_i, \psi_j\right)_{N\times N}\right]$$
$$= \exp\frac{1}{2\pi i}\int_0^\infty\int_0^{2\pi} G_z^P(t, e^{i\theta})\frac{de^{i\theta}}{e^{i\theta} - \tau}\ \frac{dt}{t - \lambda}$$

for all $\lambda \notin [0, \infty)$ and $|\tau| \neq 1$. The identity

$$\delta_z(t) = \frac{1}{2\pi}\int_0^{2\pi} G_z^P(t, e^{i\theta})d\theta \quad \text{a.e.} \tag{3}$$

is established in [1]. The function G_z^P is referred to as the polar principal function for the operator C_z. There is a second principal function G defined for the operator C. Write $C = U - iV$ where U, V are selfadjoint. There exists [1] an integrable function G on R^2 such that

$$\det\left[I_N + \left[\left((V - v)^{-1}(U - u)^{-1}\psi_i, \psi_j\right)\right]_{N\times N}\right]$$
$$= \exp\int\int G(\nu, \mu)\frac{d\nu}{\nu - v}\ \frac{d\mu}{\mu - u}.$$

The basic result relating G and G_z^P is the identity

$$G_z^P(t, e^{i\theta}) = G(\nu,\mu) \quad \text{a.e.} \tag{4}$$

where $\mu + i\nu - \bar{z} = \sqrt{t}\, e^{i\theta}$ [1].

Then

$$\int_0^\infty \frac{\delta_z(t)}{t}\, dt = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}\frac{G_z^P(t, e^{i\theta})}{t}\, dt\, d\theta$$
$$= \frac{1}{\pi}\int\int\frac{G(\nu,\mu)}{|\mu + i\nu - \bar{z}|^2}\, d\mu\, d\nu.$$

There is a subset $B \subset A = \{z = \mu + i\nu\colon G(\nu, \mu) \neq 0\}$ such that the last integral is infinite at every $\bar{z} \in B$, moreover the planar measure of $A \setminus B$ is zero.

It is known that the essential closure of A is $\sigma(C^*)$. This implies the essential closure of B is $\sigma(C^*)$. Thus whenever Δ is a disc intersecting $\sigma(C)$ there are points in Δ at which $\int_0^\infty(\delta_z(t)/t)\, dt = \infty$. Consequently, if Δ is a disc intersecting $\sigma(C)\setminus\pi_0(C)$ there are points z in Δ with

$$\det\left[I_N - \left[\left(\tilde{\psi}_i(z), \tilde{\psi}_j(z)\right)\right]_{N\times N}\right] = 0. \tag{5}$$

It will be shown that the matrix $R(z) = [(\tilde{\psi}_i(z), \tilde{\psi}_j(z))]$ satisfies $0 < R(z) < I_N$. Note first

$$\tilde{\psi}_i(z) = W_z^*(P_z + D)^{-1/2}\psi_i \qquad (1 \leq i \leq N).$$

Therefore

$$\begin{aligned}\left(\tilde{\psi}_i(z), \tilde{\psi}_j(z)\right) &= \left((P_z + D)^{-1/2}\psi_i, (P_z + D)^{-1/2}\psi_j\right)\\ &= \lim_{\lambda\to 0}\left((P_z + D - \lambda)^{-1/2}\psi_i, (P_z + D - \lambda)^{-1/2}\psi_j\right)\\ &= \lim_{\lambda\to 0}\left((P_z + D - \lambda)^{-1}\psi_i, \psi_j\right).\end{aligned}$$

The result will follow when it is established that

$$R(z, \lambda) = \left[\left((P_z + D - \lambda)^{-1}\psi_i, \psi_j\right)\right] \leqslant I_N.$$

Let $x = [x_1, \ldots, x_N] \in C^N$. Then

$$\langle R(z, \lambda)x, x\rangle = \left((P_z + D - \lambda)^{-1}\omega, \omega\right)$$

where $\omega = \sum_{i=1}^N \bar{x}_i\psi_i$. Since $(P_z + D - \lambda)^{-1} \leqslant (D - \lambda)^{-1}$ we have

$$\langle R(z, \lambda)x, x\rangle \leqslant \left((D - \lambda)^{-1}\omega, \omega\right).$$

Using the fact that $D = \sum_{i=1}^N \psi_i \otimes \psi_i$ with ψ_i orthogonal one computes

$$(D - \lambda)^{-1}\omega = \sum_{i=1}^N \frac{\bar{x}_i}{\|\psi_i\|^2 - \lambda}\,\psi_i.$$

Therefore

$$\langle R(z, \lambda)x, x\rangle \leqslant \sum_{i=1}^N \frac{|x_i|^2\|\psi_i\|^2}{\|\psi_i\|^2 - \lambda} \leqslant \sum_{i=1}^N |x_i|^2.$$

This shows $R(z, \lambda) \leqslant I_N$ and the estimate $R(z) \leqslant I_N$ follows.

Note that if $e_1, \ldots, e_N$ are orthonormal in $\mathcal{H}$, then with $A(z) = \sum_{i=1}^N (\ ,e_i)\tilde{\psi}_i(z)$, the matrix $R(z)$ is (unitarily equivalent to) the product $A^*(z)A(z)$.

Suppose now that $\Delta \cap [\sigma(C)\setminus \pi_0(C)] \neq \varnothing$ and $\tilde{\psi}_1, \ldots, \tilde{\psi}_N$ are analytic on Δ. Then from (5) it follows that the norm of $A(z)$ is achieved at $z_0 \in \Delta$. This means for some nonzero f,

$$\|A(z_0)f\| = 1 = \max_{z\in\Delta}\|A(z)\|.$$

It follows easily from the maximum modulus theorem that $A(z)f$ is constant on Δ.

This yields $\sum_{i=1}^N (f, e_i)\tilde{\psi}_i(z)$ is a constant vector g on Δ. Let $h = \sum_{i=1}^N (f, e_i)\psi_i$. Then $(C - z)g = h$ $(z \in \Delta)$ which is impossible. We conclude that one of $\tilde{\psi}_1, \ldots, \tilde{\psi}_N$ is not analytic on Δ.

The proof of Theorem 2 can now be completed as follows. Let $\Delta \cap \sigma(C) \neq \varnothing$. If $\lambda \in \Delta \cap \pi_0(C)$, then any eigenvector f_λ associated with λ satisfies $\sigma_C(f_\lambda) = \{\lambda\} \subset \Delta$. If no such λ exists then one of the weakly continuous functions $\tilde{\psi}_1, \ldots, \tilde{\psi}_N$ say $\tilde{\psi}_p$ fails to be analytic on Δ. This means that for some simple closed curve $\Gamma \subset \Delta$, $\varphi = \int_\Gamma \tilde{\psi}_p(z)\,dz \neq 0$. As in Stampfli [7] one shows $\sigma_C(\varphi) \subset \Delta$. This ends the proof of Theorem 2.

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