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SKEW-PRODUCTS WITH SIMPLE APPROXIMATIONS

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ABSTRACT. Conditions are given in order that the cartesian product of two measure-preserving invertible transformations admits an approximation. A class of skew-product transformations is defined and conditions are given for a member of this class to admit a simple approximation.

1. Preliminaries. Let $(X, \mathfrak{F}, \mu)$ be a Lebesgue space; that is, a measure space isomorphic to the unit interval with Lebesgue measure. A measure-preserving invertible point transformation of X is called an automorphism of $(X, \mathfrak{F}, \mu)$.

Let T: $X \to X$ be an automorphism. The induced automorphism T_A: $A \to A$ where $A \subset \mathfrak{F}$ is defined as follows:

$$T_A x = T^k x, \quad x \in A$$

where k is the least positive integer such that $T^k x \in A$.

Let Z denote the set of positive integers. Let f: $X \to Z$ be an integrable function. The special automorphism over T built under the function f is defined as follows:

> Put $B(k, n) = \{(x, n): x \in X, f(x) = k\}$, $n, k \in Z$ and $1 \leqslant n \leqslant k$. Put $X(f) = \bigcup_{k \geqslant 1} \bigcup_{1 \leqslant n \leqslant k} B(k, n)$. Identify X with the set $\bigcup_{k \geqslant 1} B(k, 1)$.

We may regard each set $B(k, n)$, $1 \leqslant n \leqslant k$, as a copy of $B(k, 1)$. Consequently we may extend the measure μ to $X(f)$ and form a normalised measure μ' on $X(f)$ in the obvious way. We define T_f, the special automorphism over T, by

$$T_f(x, n) = (x, n + 1) \quad \text{if } 1 \leqslant n < f(x),$$

$$T_f(x, f(x)) = (Tx, 1).$$

The following definitions are due to Katok and Stepin [4] and Chacon [1] respectively.

DEFINITION 1. An automorphism T is said to admit a cyclic approximation by periodic transformations of the first kind (a.p.t.I) with speed $f(n)$, where $f(n)$ is a sequence of real numbers decreasing to zero, if there exists a sequence of partitions $\{\xi(n)\}$, $\xi(n) = \{C_i(n): 1 \leqslant i \leqslant q(n)\}$ such that:

1. $\xi(n) \to \varepsilon_X$;
2. $\sum_{i=1}^{q(n)} \mu(TC_i(n) \Delta C_{i+1}(n)) < f(q(n))$, where $C_{q(n)+1}(n)$ means $C_1(n)$.

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DEFINITION 2. An automorphism T admits a simple approximation if there exists a sequence of partitions $\{\xi(n)\}$, $\xi(n) = \{C_i(n)\colon 1 \leqslant i \leqslant q(n)\}$ such that:

1. $\xi(n) \to \varepsilon_X$;
2. $TC_i(n) = C_{i+1}(n)$, $1 \leqslant i \leqslant q(n) - 1$.

Let R denote the positive reals.

We also require a slight adaption of Definition 1, as follows.

DEFINITION 3. An automorphism T admits a cyclic a.p.t.I with speed $f(x)$, if T admits a cyclic a.p.t.I with speed $f(n)$, where $f\colon R \to R$ and $f(x) \to 0$ as $x \to \infty$.

In **[6]** the following two results were shown.

THEOREM 1. *Let $T\colon X \to X$ admit a cyclic a.p.t.I with speed $f(n) = o(1/n^2)$ with respect to a sequence of partitions $\xi(n)$, $\xi(n)$ having $q(n)$ elements. Let $A \in \mathfrak{F}$ be approximated by sets $A(n) \leqslant \xi(n)$ with $A(n) \subset A$ such that $\mu(A \setminus A(n)) = o(1/q(n))$. Then T_A admits a simple approximation.*

THEOREM 2. *Let $T\colon X \to X$ admit a cyclic a.p.t.I with speed $g(x) = o(1/x^k)$ with respect to a sequence of partitions $\{\xi(n)\}$, $\xi(n) = \{C_i(n)\colon 1 \leqslant i \leqslant q(n)\}$. Let $f\colon X \to Z$ be integrable with $f(C_i(n)) = k_i(n) \in Z$, $1 \leqslant i \leqslant q(n)$, $n \leqslant 1$. Then T_f admits a cyclic a.p.t.I with speed $G(n) = o(1/n^k)$.*

2. Approximation of products. Let $(X, \mathfrak{F}, \mu)$ and $(Y, \mathfrak{D}, \nu)$ be Lebesgue spaces. Let S and T be automorphisms of X and Y respectively. Let $\xi(n) = \{C_i(n)\colon 1 \leqslant i \leqslant p(n)\}$ and $\zeta(n) = \{D_j(n)\colon 1 \leqslant j \leqslant q(n)\}$ be partitions in X and Y respectively. Define S_n and T_n by

$$S_n C_i(n) = C_{i+1}(n), \quad 1 \leqslant i \leqslant p(n), \text{ where } C_{p(n)+1}(n)$$

means $C_1(n)$, and

$$T_n D_j(n) = D_{j+1}(n), \quad 1 \leqslant j \leqslant q(n), \text{ where } D_{q(n)+1}(n)$$

means $D_1(n)$.

The lemma below is easily verified.

LEMMA 1. *If $(p(n), q(n)) = 1$, then $S_n \times T_n$ maps the elements of $\xi(n) \times \zeta(n)$ cyclically; that is*

$$S_n \times T_n\big(C_i(n) \times D_j(n)\big) = C_{i+1}(n) \times D_{j+1}(n)$$

and $(S_n \times T_n)^k(C_i(n) \times D_j(n)) \neq C_i(n) \times D_j(n)$ for $k < p(n)q(n)$.

The theorem which follows gives the conditions for a product of automorphisms to admit a cyclic a.p.t.I with speed of the form $O(1/n^k)$.

THEOREM 3. *Let $S\colon X \to X$ admit a cyclic a.p.t.I with speed $f(n)$ $O(1/n^{(r+1)k})$ where $r \geqslant 2$, $k \geqslant 1$, $r, k \in Z$, with respect to a sequence of partitions $\{\xi(n)\}$, $\xi(n) = \{C_i(n)\colon 1 \leqslant i \leqslant p(n)\}$. Let $T\colon Y \to Y$ admit a cyclic a.p.t.I with speed $g(n) = O(1/n^{(r+1)k})$, with respect to a sequence of partitions $\{\zeta(n)\}$, $\zeta(n) = \{D_j(n)\colon 1 \leqslant j \leqslant q(n)\}$. Suppose that:*

1. $(p(n), q(n)) = 1$;
2. $p(n) < q(n)$;
3. $p(n)^r > q(n)$.

Then $S \times T$ admits a cyclic a.p.t.I with speed $G(n) = 2\delta/n^k$, for some $\delta > 0$.

PROOF.

$$\sum_{i=1}^{p(n)} \sum_{j=1}^{q(n)} \mu \times \nu\big(S \times T(C_i(n) \times D_j(n))\Delta(C_{i+1}(n) \times D_{j+1}(n))\big)$$

$$\leqslant \sum_i \sum_j \mu(C_i(n))\nu(TD_j(n)\Delta D_{j+1}(n)) + \nu(D_j(n))\,\mu(SC_i(n)\Delta C_{i+1}(n))$$

$$\leqslant p(n)\,\mu(C_1(n)) \sum_j \nu(TD_j(n)\Delta D_{j+1}(n))$$

$$+ q(n)\nu(D_1(n)) \sum_i \mu(SC_i(n)\Delta C_{i+1}(n))$$

$$\leqslant \sum_j \nu(TD_j(n)\Delta D_{j+1}(n)) + \sum_i \mu(SC_i(n)\Delta C_{i+1}(n))$$

$$< f(p(n)) + g(q(n)).$$

Since $f(n) = O(1/n^{(r+1)k})$, $f(n) < \delta_1/n^{(r+1)k}$ for some $\delta_1 > 0$. Similarly, $g(n) < \delta_2/n^{(r+1)k}$ for some $\delta_2 > 0$. Put $\delta = \max\{\delta_1, \delta_2\}$ then

$$f(p(n)) + g(q(n)) < \delta/p(n)^{(r+1)k} + \delta/q(n)^{(r+1)k} < 2\delta/p(n)^k q(n)^k,$$

since by (3) $p(n)^r > q(n)$.

Put $G(n) = 2\delta/n^k$. By Lemma 1 and the above, $\{\xi(n) \times \zeta(n)\}$ is a sequence of partitions with respect to which $S \times T$ admits a cyclic a.p.t.I with speed $g(n)$.

In [1] it is shown that if an automorphism admits a cyclic a.p.t.I with speed θ/n, $\theta < 1$, then it has simple spectrum. Consequently if $\delta < \frac{1}{2}$ and $k \geqslant 1$, then $S \times T$ will have simple spectrum. If $f(n) = o(1/n^{(r+1)k})$ and $g(n) = o(1/n^{(r+1)k})$ then it is easily seen that $S \times T$ will have speed of approximation $G(n) = 2\delta/n^k$ for any $\delta > 0$.

If S and T and $S \times T$ all have simple spectrum then S and T can have no common spectral type and consequently by a result of Hahn and Parry [3] S and T are disjoint.

If T has simple spectrum then $T \times T$ has spectral multiplicity strictly greater than one. Consequently Theorem 3 shows that there are restrictions on the types of approximating partitions which exist for T, when the speed of approximation is of the order of $O(1/n^3)$.

In a similar way to Theorem 3 we can also show the following.

THEOREM 4. *Let S: $X \to X$ admit a cyclic a.p.t.I with speed $f(n) = a/\log n$, $a > 0$, with respect to a sequence of partitions $\{\xi(n)\}$, $\xi(n)$ having $p(n)$ elements. Let T: $Y \to Y$ admit a cyclic a.p.t.I with speed $g(n) = b/\log n$, $b > 0$, with* *respect to a sequence of partitions* $\{\zeta(n)\}$, $\zeta(n)$ *having* $q(n)$ *elements. Suppose* $p(n)$ *and* $q(n)$ *satisfy*:

1. $(p(n), q(n)) = 1$;
2. $p(n) < q(n)$;
3. $\log q(n) < k \log p(n)$.

Then $S \times T$ *admits a cyclic a.p.t.I with speed* $g(n) = (k+1)(a+b)/\log n$.

3. Skew-products with simple approximations. The class of skew-products we shall consider were discussed by Newton in [5] where formulae were given for calculating their entropy. Goodson has considered skew-products of a different type in [2]. He has given conditions for finite skew-products to admit a simple approximation.

Let S and T be automorphisms of X and Y respectively. Let f: $X \to Z$ be integrable. The skew-products considered below are of the form

$$\psi(x,y) = (Sx, T^{f(x)}y), \quad x \in X, y \in Y.$$

Let η be the measure which assigns measure 1 to each point of Z. Let V be the subset of $X \times Y \times Z$ defined by $(x, y, i) \in V$ if $i \leqslant f(x)$. So that $V = V' \times Y$ where V' is the subset of $X \times Z$ defined by $(x, i) \in V'$ if $i \leqslant f(x)$. It is easily seen that

$$\mu \times \nu \times \eta(V) = \mu \times \eta(V') = \int f(x) d\mu.$$

We can consider V as a Lebesgue space with normalised measure μ' defined by

$$\mu'(A) = \mu \times \nu \times \eta(A) \cdot \left(\int f(x) d\mu\right)^{-1},$$

where $A \subset V$.

Define an automorphism ϕ on V by

$$\begin{aligned} \phi(x, y, i) &= (x, Ty, i+1) \quad && \text{if } i < f(x), \\ &= (Sx, Ty, 1) && \text{if } i = f(x). \end{aligned}$$

Then $\phi = S_f \times T$. Furthermore it is clear that ψ is the automorphism induced by ϕ on the set $X \times Y \times \{1\}$.

Definition 4. Let ξ be a partition in X such that every element of ξ is contained in exactly one of the sets $B(k, 1)$, defined in the first section, for some k. Order the sets $B(k, n)$, $k \geqslant 1$, $1 \leqslant n \leqslant k$, lexicographically. Then ξ^f is the partition in $X(f)$ consisting of the elements $C \in \xi$, together with for each $C \in \xi$, where $C \subset B(k, 1)$, a copy of C in each of the sets $B(k, n)$, $1 \leqslant n \leqslant k$. The ordering on ξ^f is that inherited from the sets $B(k, n)$.

We now give conditions in order that ψ should admit a simple approximation.

Theorem 5. *Let* S: $X \to X$ *admit a cyclic a.p.t.I with speed* $h(x) = O(1/x^{3(r+1)})$, $r \geqslant 2$, $r \in Z$, *with respect to a sequence of partitions* $\{\xi(n)\}$,

$\xi(n) = \{C_i(n)\colon 1 \leqslant i \leqslant s(n)\}$. *Let* $f\colon X \to Z$ *be integrable with* $f(C_i(n)) = k_i(n)$, $1 \leqslant i \leqslant s(n)$, $n \geqslant 1$, *and suppose that* $\xi^f(n)$ *has* $p(n)$ *elements. Let* $T\colon Y \to Y$ *admit a cyclic a.p.t.I with speed* $g(n) = O(1/n^{3(r+1)})$ *with respect to a sequence of partitions* $\{\zeta(n)\}$, $\zeta(n) = \{D_j(n)\colon 1 \leqslant j \leqslant q(n)\}$. *Suppose that*:

1. $(p(n), q(n)) = 1$;
2. $p(n) < q(n)$;
3. $p(n)^r > q(n)$;
4. $s(n)^{r+1}\mu(X \setminus \bigcup_{i=1}^{s(n)} C_i(n)) \to 0$ *as* $n \to \infty$;
5. $q(n)^2\nu(Y \setminus \bigcup_{j=1}^{q(n)} D_j(n)) \to 0$ *as* $n \to \infty$.

Then $\psi(x,y) = (Sx, T^{f(x)}y)$ *admits a simple approximation.*

PROOF. By Theorem 2, S_f admits a cyclic a.p.t.I with speed $H(n) = O(1/n^{3(r+1)})$, with respect to the sequence of partitions $\xi^f(n)$.

By the remarks following Theorem 3, ϕ admits a cyclic a.p.t.I with speed $G(n) = O(1/n^2)$ with respect to the sequence of partitions $\{\xi^f(n) \times \zeta(n)\}$. Now

$$p(n)q(n)\,\mu'\left[X \times Y \times \{1\} \setminus \sum_{i=1}^{s(n)} \sum_{j=1}^{q(n)} C_i(n) \times D_j(n) \times \{1\}\right]$$

$$\leqslant p(n)q(n)\,\mu\left(X \setminus \sum_{i=1}^{s(n)} C_i(n)\right) + p(n)q(n)\nu\left[Y \setminus \sum_{j=1}^{q(n)} D_j(n)\right]$$

$$\leqslant p(n)^{r+1}\mu\left(X \setminus \bigcup_i C_i(n)\right) q(n^2)\nu\left(Y \setminus \bigcup_j D_j(n)\right) \to 0 \text{ as } n \to \infty,$$

since $p(n) \leqslant s(n).\ (1 + 2\int f\,d\mu)$ for n sufficiently large.

Hence $\phi_{X \times Y \times \{1\}}$ admits a simple approximation by Theorem 1, which completes the proof.

It is fairly easy to manufacture examples of automorphisms S and T which satisfy the conditions of Theorem 5 by the stacking method. Using methods similar to those in [**4**], we can use continued-fraction theory to provide rotations of the unit interval, S and T, which satisfy the conditions of Theorem 3. As a consequence of this, it is possible to give examples of skew-products, of the type discussed above, with interval exchange transformations in the base, and rotations in the fibres, which have simple spectrum, without using Theorem 5.

Chacon [**1**] has generalised the idea of cyclic a.p.t.I to that of approximation with multiplicity N. We remark that all the results shown above have straightforward generalisations to the 'multiplicity N situation'. We then have the following generalisation of Theorem 5.

THEOREM 6. *Let* S, f *and* T *be as in Theorem* 5. *Suppose that*:

1. $g.c.d.\ (p(n), q(n)) = N$;
2. $p(n) < q(n)$;

3. $p(n)^r > q(n)$;
4. $s(n)^{r+1}\mu(X \setminus \bigcup_{i=1}^{s(n)} C_i(n)) \to 0$ *as* $n \to \infty$;
5. $q(n)^2\nu(Y \setminus \bigcup_{j=1}^{q(n)} D_j(n)) \to 0$ *as* $n \to \infty$.

Then $\psi(x, y) = (Sx, T^{f(x)}y)$ *admits a simple approximation with multiplicity* N.

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