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ON THE CENTRAL LIMIT THEOREM IN F-SPACES

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ABSTRACT. In this note we improve the known result on the integrability of F-norms with respect to Gaussian measures and obtain a Central Limit Theorem for probability measures on arbitrary separable F-spaces.

1. Let E be a real linear space. A function $\|\cdot\|$: $E \to [0, \infty)$ is called F-seminorm if it satisfies

(i) $\|x + y\| \leqslant \|x\| + \|y\|$, $x, y \in E$,

(ii) $\|\alpha x\| \leqslant \|x\|$, $\alpha \in \mathbf{R}$, $|\alpha| \leqslant 1$, $x \in E$,

(iii) $\|\alpha x\| \to 0$, $\alpha \in \mathbf{R}$, $\alpha \to 0$, $x \in E$.

$\|\cdot\|$ is an F-norm if $\|x\| = 0$ implies $x = 0$ and it is called p-homogeneous, $0 < p \leqslant 1$, if $\|\alpha x\| = |\alpha|^p \|x\|$ for $\alpha \in \mathbf{R}$, $x \in E$. Every F-norm defines a translation invariant metric $d(x, y) = \|x - y\|$ which makes E into a linear metric space. If E is complete with respect to $d(x, y)$ we call it an F-space. We will consider separable F-spaces only. Provided with its Borel σ-algebra $\mathfrak{B}$, $(E, \mathfrak{B})$ then becomes a measurable linear space in the sense of Fernique [2], whose definition of a Gaussian measure we adopt. Let the mappings $T_{s,t}$: $E \times E \to E \times E$ be defined as

$$T_{s,t}(x, y) = (sx - ty, tx + sy), \qquad s, t \in \mathbf{R}.$$

We say that a probability measure $\mathfrak{S}$ on E is mean zero Gaussian if the product measure $\mathfrak{S} \otimes \mathfrak{S}$ is invariant with respect to the family $\{T_{s,t}: s^2 + t^2 = 1\}$.

Concerning the integrability of F-norms with respect to Gaussian measures we have the following fundamental result.

THEOREM 1. *Let $(E, \|\cdot\|)$ be a separable F-space with p-homogeneous F-norm, $0 < p \leqslant 1$, let $r = 2/(2 - p)$ and $\mathfrak{S}$ be a mean zero Gaussian measure on $(E, \mathfrak{B})$. Then there exists an $\varepsilon > 0$ such that*

$$\int \exp(\varepsilon \|x\|^r)\mathfrak{S}(dx) < \infty.$$

We omit the proof since it follows from a careful inspection and adaptation of Fernique's original proof [2] as indicated in [4] for $r = 1$. We only remark that the theorem still holds in the more general situation, when $(E, \mathfrak{B})$ is just a measurable linear space and $\|\cdot\|$ a measurable pseudo-F-seminorm, which satisfies $\mathfrak{S}(x: \|x\| < \infty) > 0$.

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2. If E^*, the topological dual of E, separates points of E, our definition of Gaussian measures coincides with the usual one: $\mathfrak{S}$ is mean zero Gaussian if and only if every $f \in E^*$ is a (possibly degenerate) normally distributed real random variable with zero mean on $(E, \mathfrak{B}, \mathfrak{S})$. In this case we assume throughout all probability measures μ to be weakly centered, i.e.

$$\int f(x)\,\mu(dx) = 0, \qquad f \in E^*.$$

If μ has weak second moment, i.e. $E^* \subseteq L^2(E, \mathfrak{B}, \mu)$, then

$$\Gamma_\mu(f, g) = \int f(x) g(x)\,\mu(dx), \qquad f, g \in E^*$$

will denote the covariance of μ, and we call μ pregaussian if it has the same covariance as some Gaussian measure $\mathfrak{S}$. By $\mathfrak{L}(X)$ we denote the distribution of a random element X defined on some probability space with values in E. Finally we say that μ satisfies the Central Limit Theorem (CLT) if the distributions of $\sqrt{n}^{\,-1}(X^{(1)} + \cdots + X^{(n)})$ converge $\|\cdot\|$-weakly to a Gaussian measure $\mathfrak{S}$ on E, where $X^{(j)}$ are independent copies of X, $\mathfrak{L}(X) = \mu$.

In [3] we had investigated Gaussian measures and the CLT in certain Orlicz sequence spaces determined by a sequence of subadditive Orlicz functions as follows

$$l_\Phi = \left\{(y_j)\colon y_j \in \mathbf{R}, \sum_j \varphi_j(|y_j|) < \infty\right\}, \qquad \|y\|_\Phi = \sum_j \varphi_j(|y_j|).$$

In the study of measures on these sequence spaces a basic role is played by the so called "standard deviation vector" $\sigma(\mu)$ defined coordinatewise as

$$(\sigma(\mu))_j = \Gamma_\mu(e_j^*, e_j^*)^{1/2} = \left(\int y_j^2\,\mu(dy)\right)^{1/2}, \quad j = 1, 2, \ldots,$$

$\{e_j, e_j^*\}$ denoting the canonical basis of l_Φ.

For our present purpose the results of [3] may be summarized as follows.

THEOREM 2. (i) *A probability measure μ on l_Φ is pregaussian if and only if $\sigma(\mu) \in l_\Phi$.*

(ii) *A probability measure μ on l_Φ satisfies the CLT if and only if it is pregaussian.*

Further results emphasizing the importance of the standard deviation vector will be published elsewhere.

Now we give a sufficient condition for the CLT to hold in an arbitrary separable F-space $(E, \|\cdot\|)$.

THEOREM 3. *Let $X = \sum_i \xi_i(\omega) x_i$, $\sigma \neq x_i \in E$, ξ_i real random variables, be a.s. absolutely convergent, $\mu = \mathfrak{L}(X)$. If the variables ξ_i have zero expectations, finite second moment and*

$$\sum_i \|\sigma(\xi_i)x_i\| < \infty, \tag{$*$}$$

then μ satisfies the Central Limit Theorem in E.

PROOF. Setting $\varphi_j(t) = \|tx_j\|$, $t > 0$, $j = 1, 2, \ldots,$ we obtain subadditive Orlicz functions in the above sense. Denoting by l_Φ the corresponding Orlicz sequence space

$$v\colon l_\Phi \to E, \quad v((\eta_j)) = \sum_j \eta_j x_j$$

becomes a linear and continuous map. Now $\xi(\omega) = (\xi_1(\omega), \xi_2(\omega), \ldots)$ defines a random element with values in l_Φ which is pregaussian and satisfies the CLT in l_Φ because of Theorem 2 and condition ($*$). Hence there exists a Gaussian measure $\mathfrak{S}$ on l_Φ such that $\theta_n(\nu^{n^*})$ converges $\|\cdot\|_\Phi$-weakly to $\mathfrak{S}$, where $\theta_n x = (1/\sqrt{n})x$, ν is the distribution of ξ, ν^{n^*} its n-fold convolution. Since v is continuous, it follows that $v(\mathfrak{S})$ is Gaussian on E and that $\theta_n(\mu^{n^*})$ converges $\|\cdot\|$-weakly to $v(\mathfrak{S})$.

One should remark that we did not assume the variables ξ_i to be independent. Moreover, choosing the x_i to form a countable dense subset of E, the corresponding map v becomes onto [5] and it follows that every probability measure μ on E can be obtained as $\mu = \mathfrak{L}(X)$, X absolutely convergent as presumed. However, due to its general nature, condition ($*$) is quite strong in many spaces. But according to Theorem 2, if $x_j = e_j$, it is necessary in all sequence spaces l_Φ, including the Banach space l^1. Hence condition ($*$) cannot be weakened without restricting the class of spaces for which Theorem 3 is valid. As far as only Banach spaces are concerned, Theorem 3 could of course be obtained using [**1**, Theorem 6.1] instead of Theorem 2.

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REFERENCES

1. A. D. de Acosta, *Existence and convergence of probability measures in Banach spaces*, Trans. Amer. Math. Soc. **152** (1970), 273–298.

2. X. Fernique, *Intégrabilité des vecteurs gaussiens*, C. R. Acad. Sci. Paris **270** (1970), 1698–1699.

3. W. E. Helm, *Gaussian random elements in certain Orlicz sequence spaces*, Bull. Acad. Polon. Sci. **25** (1977), 507–514.

4. T. Inglot and A. Weron, *On Gaussian random elements in some non-Banach spaces*, Bull. Acad. Polon. Sci. **22** (1974), 1039–1043.

5. Ph. Turpin, *Convexités dans les espaces vectoriels topologiques généraux*, Thèse, Orsay, 1974.

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