

## METRIZABILITY OF COMPACT SETS AND CONTINUOUS SELECTIONS<sup>1</sup>

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**ABSTRACT.** It is shown that, in two of Michael's theorems on continuous selections, the condition that the range of the correspondence under consideration be metrizable is not only essential (as known through several counterexamples), but in some sense also necessary. This yields a characterization of metrizability for compact spaces and compact convex sets by means of continuous selections.

**1. Introduction.** We consider the following statement about compact sets which comprises special cases of two selection theorems due to Michael [7, Theorem 1.2] and [8, Theorem 1.2]:

A compact space (a compact convex set)  $K$  which is metrizable has the following property:

If  $X$  is a compact zerodimensional (a compact) space, then every lower semicontinuous correspondence  $\Phi$  between  $X$  and  $K$  such that  $\Phi(x)$  is closed (closed and convex) for all  $x$  admits a continuous selection.

It is known—e.g. from counterexamples given by Corson and Lindenstrauss [1] and von Weizsäcker [10]—that metrizability is an essential condition for  $K$  to have the above property. We shall show that this is even a necessary condition, i.e. that  $K$  must be metrizable if it has this property. As a consequence, we get that—similar to Michael's characterization of paracompactness for  $T_1$ -spaces—metrizability of compact spaces and compact convex sets may be characterized through continuous selections.

Our main tools will be some results of Efimov [3] about dyadic spaces and a suitable modification of von Weizsäcker's already mentioned example.

**2. Notation.** All topological spaces are assumed to be Hausdorff in this paper.

A *correspondence*  $\Phi$  between two sets  $X$  and  $Y$  is a subset of  $X \times Y$  such that  $\Phi(x) := \{y \in Y: (x, y) \in \Phi\}$  is nonempty for all  $x \in X$  (since such a  $\Phi$  can be identified with a map from  $X$  into the nonempty subsets of  $Y$ , correspondences are often called set, or multivalued maps in the literature). A *selection* for  $\Phi$  is a map  $\phi: X \rightarrow Y$  such that  $\phi(x) \in \Phi(x)$  for all  $x$ . If  $X$  and  $Y$  are also topological spaces,  $\Phi$  is called *lower semicontinuous* (l.s.c) if  $\{x \in X:$

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$\Phi(x) \cap U \neq \emptyset$  is open for all open  $U \subseteq Y$ .

For a set  $M$  and a positive integer  $n$  we denote by  $M(n)$  the set of nonempty subsets of  $M$  with at most  $n$  elements. If  $M$  is also a topological space,  $M(n)$  is always assumed to carry the *Vietoris topology* (also called exponential or finite topology). It is generated by the sets  $\{A \in M(n): A \subset U\}$  and  $\{A \in M(n): A \cap U \neq \emptyset\}$ , where  $U$  runs through the open subsets of  $M$ .

A *dyadic space* is a compact space which is the continuous image of some Cantor cube.

By a *compact convex set* we always mean a compact convex subset of some (real or complex) locally convex space.

Finally, if  $A$  is a subset of a linear space  $E$  we denote by  $\text{conv } A$  its convex hull and call a map  $\phi: A(n) \rightarrow E$  a *convex selection* if  $\phi(B) \in \text{conv } B$  holds for all  $B \in A(n)$ .

### 3. The theorems.

**THEOREM 1.** *The following properties of a compact space  $K$  are equivalent:*

(M)  $K$  is metrizable.

(S1) *If  $X$  is a zerodimensional compact space, then every l.s.c. correspondence  $\Phi$  between  $X$  and  $K$  such that  $\Phi(x)$  is closed for all  $x$  admits a continuous selection.*

**THEOREM 2.** *The following properties of a compact convex set  $K$  are equivalent:*

(M)  $K$  is metrizable.

(S2) *If  $X$  is a compact space, then every l.s.c. correspondence  $\Phi$  between  $X$  and  $K$  such that  $\Phi(x)$  is closed and convex for all  $x$  admits a continuous selection.*

Before we start proving these theorems, let us make the following remark:

**REMARK 1.** Given topological spaces  $X$  and  $Y$ ,  $A \subseteq X$  closed and  $f: A \rightarrow Y$  continuous, the correspondence  $\Phi_f$  between  $X$  and  $Y$  defined by  $\Phi_f(x) := \{f(x)\}$  if  $x \in A$ ,  $\Phi_f(x) := Y$  if  $x \notin A$  is l.s.c. [6, Example 1.3] and a continuous selection for  $\Phi_f$  is nothing but a continuous extension of  $f$ . This shows that compact spaces with property (S1) and compact convex sets with property (S2) are absolute extensors for compact zerodimensional and compact spaces, respectively.

Since arbitrary products of compact intervals are absolute extensors for normal spaces (cf. [2, Chapter VII, Corollary 5.2]) our theorems show in particular that for some extension theorems there is no corresponding selection theorem, although this is often the case (cf. [6]).

**4. Proof of Theorem 1.** We only have to show that (S1) implies (M). By a theorem of Alexandroff (cf. [8, Proposition 8.3.5]) there exists a Cantor cube  $C$ ,  $A \subseteq C$  closed and  $f: A \rightarrow K$  continuous and onto. By Remark 1,  $f$  can be

extended to a continuous map from  $C$  onto  $K$ , hence  $K$  is dyadic. Since each closed subspace of  $K$  has property (S1), too, it follows that  $K$  is hereditarily dyadic and therefore metrizable [3, Theorem 4].

**5. Proof of Theorem 2.** We begin with some preliminary results. Lemma 1 and Lemma 2 below are due to von Weizsäcker [10, Proof of Theorem B.1, parts 1–3] and stated without proof. The proposition is a modification of Theorem B.1 in [10] and will be proved—with the aid of Lemma 3—using almost the same arguments von Weizsäcker used.

**LEMMA 1.** *Let  $T$  be an uncountable set and  $\tau: T(2) \rightarrow T$  be a map such that  $\tau(A) \in A$  holds for all  $A \in T(2)$ . Then there exists  $x \in T$  such that  $\{y \in T: \tau(\{x, y\}) = x\}$  is infinite.*

**LEMMA 2.** *Let  $X = I \cup \{\omega\}$ , ( $\omega \notin I$ ) be a one-point compactification of an uncountable discrete space  $I$  and suppose that for each  $x \in X$  a map  $h^x: X \rightarrow [0, 1]$  is given such that:*

- (i)  $h^x(y) = 1 - h^y(x)$  for all  $x, y \in X$ .
- (ii)  $h^x$  is continuous at  $\omega$  for all  $x \in I$ .

*Then  $\{y \in I: |h^y(\omega) - 1/2| > \varepsilon\}$  is finite for all  $\varepsilon > 0$ .*

**LEMMA 3.** *Let  $E$  be a topological linear space,  $X \subseteq E$ ,  $x_0 \in X$ ,  $n \in \{2, 3\}$  and  $\gamma: X(n) \rightarrow E$  be a convex selection which is continuous at  $\{x_0\}$ .*

*Then for each  $x \in X$  there exists a map  $f^x: X(n-1) \rightarrow [0, 1]$  with  $f^x(\{x\}) = 1/2$  and a convex selection  $\gamma^x: X(n-1) \rightarrow E$  such that*

$$\gamma(A \cup \{x\}) = f^x(A)x + (1 - f^x(A))\gamma^x(A)$$

*for all  $A \in X(n-1)$ .*

*Furthermore,  $f^x$  is continuous at  $\{x_0\}$  if  $x \neq x_0$ , and  $\gamma^x$  is continuous at  $\{x_0\}$  if  $x \neq \gamma(\{x, x_0\}) \neq x_0$ .*

**PROOF.**  $f^x$  and  $\gamma^x$  obviously exist for all  $x \in X$ .

Now let  $x \neq x_0$ ,  $\alpha := f^x(\{x_0\})$  and  $\varepsilon > 0$ . Choose a continuous real linear form  $l$  on  $E$  such that  $l(x_0 - x) = 1$ . Since  $U := \{A \in X(n-1): |l(x_0 - y)| < \varepsilon/4 \text{ for all } y \in A\}$  is a neighborhood of  $\{x_0\}$  and since  $L: X(n-1) \rightarrow \mathbf{R}$  defined by  $L(A) := l(\gamma(A \cup \{x\}))$  is continuous at  $\{x_0\}$  we can choose a neighborhood  $V \subseteq U$  of  $\{x_0\}$  such that  $|L(\{x_0\}) - L(A)| < \varepsilon/4$  for all  $A \in V$ .

Let  $A \in V$  and suppose that  $A = \{y, z\}$  where  $y$  and  $z$  are possibly equal. Then there exist positive numbers  $\lambda, \mu$  such that  $f^x(A) + \lambda + \mu = 1$  and  $\gamma(A \cup \{x\}) = f^x(A)x + \lambda y + \mu z$ . We now get

$$\begin{aligned} \varepsilon/4 &> |L(\{x_0\}) - L(A)| \\ &= |l(\alpha x + (1 - \alpha)x_0 - f^x(A)x - \lambda y - \mu z)| \\ &= |1 - \alpha - \lambda l(y - x) - \mu l(z - x)| \\ &\geq ||1 - \alpha - (\lambda + \mu)l(y - x)| - \mu|l(z - y)||. \end{aligned}$$

Since  $0 < \mu \leq 1$  and  $|l(z - y)| < \varepsilon/2$  this implies

$$|1 - \alpha - (\lambda + \mu)l(y - x)| < (3\varepsilon)/4.$$

Because  $0 < \lambda + \mu \leq 1$  and  $|l(y - x) - 1| = |l(y - x) - l(x_0 - x)| < \varepsilon$  we finally get

$$|\alpha - f^x(A)| = |1 - \alpha - \lambda - \mu| < (3\varepsilon)/4 + \varepsilon/4 = \varepsilon$$

from which the continuity of  $f^x$  at  $\{x_0\}$  follows.

Let now  $x \in X$  be such that  $x \neq \gamma(\{x, x_0\}) \neq x_0$ . This implies  $x \neq x_0$  and  $0 < f^x(\{x_0\}) < 1$ . Then  $f^x$  is continuous at  $\{x_0\}$  and we therefore may choose a neighborhood  $W$  of  $\{x_0\}$  on which the functions  $1/f^x$  and  $1/(1 - f^x)$  are well defined and continuous at  $\{x_0\}$ . Since  $\gamma^x(A) = 1/(1 - f^x(A))[\gamma(A \cup \{x\}) - f^x(A)x]$  holds for  $A \in W$ , the continuity of  $\gamma^x$  at  $\{x_0\}$  now follows.

**PROPOSITION.** *Let  $E$  be a topological linear space,  $I \subseteq E$  an uncountable set and  $\omega \in E \setminus I$ . Suppose that:*

(i)  *$I$  is discrete in the relative topology and  $X := I \cup \{\omega\}$  is a one-point compactification of  $I$ .*

(ii) *For  $x \neq y$  in  $I$  the set  $\{x, y, \omega\}$  is affinely independent.*

*Then there exists no continuous convex selection  $\phi: X(3) \rightarrow E$ .*

**PROOF.** Assume, if possible, that  $\phi: X(3) \rightarrow E$  is a continuous convex selection. For  $\phi$  restricted to  $X(2)$  and  $x \in X$  we choose  $g^x: X \rightarrow [0, 1]$  according to Lemma 3 (we identify  $X$  and  $X(1)$ ). Then by Lemma 2 the set  $M' := \{y \in I: g^y(\omega) \neq 1/2\}$  is at most countable, hence  $M := \{x \in I: \phi(\{x, \omega\}) = \frac{1}{2}(x + \omega)\}$  is uncountable. Let  $\varepsilon > 0$  be such that  $\varepsilon + \varepsilon^2 \leq 1/16$  and  $x \in M$ . For  $\phi$  choose  $f^x: X(2) \rightarrow [0, 1]$  and a convex selection  $\phi^x: X(2) \rightarrow E$  according to Lemma 3. Then  $f^x$  and  $\phi^x$  are continuous at  $\{\omega\}$  since  $\phi(\{x, \omega\}) = \frac{1}{2}(x + \omega)$  and by definition of the topologies on  $X$  and  $X(2)$  there exists  $M(x) \subseteq X$  such that

(1)  $x \notin M(x)$ ,  $\omega \in M(x)$  and  $X \setminus M(x)$  is finite.

(2)  $|f^x(A) - 1/2| = |f^x(A) - f^x(\{\omega\})| < \varepsilon$  for  $A \subseteq M(x)$ .

We now apply Lemma 3 to  $\phi^x$  and choose for each  $z \in X$  a map  $g_x^z: X \rightarrow [0, 1]$  according to that lemma. Again by Lemma 2 the set  $N'(x) := \{z \in I: |g_x^z(\omega) - 1/2| > \varepsilon\}$  is finite. Consequently, we have for  $N(x) := M(x) \setminus N'(x)$ :

(3)  $M(x) \setminus N(x)$  is finite.

(4)  $|g_x^y(\omega) - 1/2| \leq \varepsilon$  for all  $y \in N(x)$ .

Let now  $N(x)$  be chosen this way for all  $x \in M$ . Then either  $x \notin N(y)$  or  $y \notin N(x)$  holds for all  $x, y \in M$ . For assume that  $x \in N(y)$  and  $y \in N(x)$  for some  $x$  and  $y$  in  $M$ . Then  $\{x, y, \omega\}$  is affinely independent by (1) and (ii), therefore the equation

$$\begin{aligned}
 & f^x(\{y, \omega\})x + (1 - f^x(\{y, \omega\})) [g_x^y(\omega)y + (1 - g_x^y(\omega))\omega] \\
 &= f^x(\{y, \omega\})x + (1 - f^x(\{y, \omega\}))\phi^x(\{y, \omega\}) = \phi(\{x, y, \omega\}) \\
 &= f^y(\{x, \omega\})y + (1 - f^y(\{x, \omega\})) [g_y^x(\omega)x + (1 - g_y^x(\omega))\omega]
 \end{aligned}$$

shows that  $f^x(\{y, \omega\}) = (1 - f^y(\{x, \omega\}))g_y^x(\omega)$  holds. But this is impossible because (2) implies  $|f^x(\{y, \omega\}) - 1/2| \leq 1/16$ , while (2) and (4) imply  $|(1 - f^y(\{x, \omega\}))g_y^x(\omega) - 1/4| \leq 1/16$ .

So we can choose a map  $\tau: M(2) \rightarrow M$  such that  $\tau(A) \in A$  for all  $A \in M(2)$  and such that  $y \notin N(x)$  if  $\tau(\{x, y\}) = x$ . Then by Lemma 1 there exists  $x \in M$  such that  $\{y \in M: \tau(\{x, y\}) = x\}$  is infinite. But by the choice of  $\tau$  this set is contained in  $M \setminus N(x)$  which is finite by (1) and (3).

This final contradiction shows that  $\phi$  cannot exist which proves our proposition.

We are now ready to prove Theorem 2.

Again, we only have to show that (S2) implies (M). To this end, assume that  $K$  has property (S2) and is *not* metrizable. First we see as in the proof of Theorem 1 that  $K$  is dyadic because it has property (S2). Then by another result of Efimov [3, p. 499, Proof of Theorem 4] there exists an uncountable discrete set  $T \subseteq K$  and  $\omega \in K \setminus T$  such that  $T \cup \{\omega\}$  is a one-point compactification of  $T$  because  $K$  is not metrizable. Since for each one-dimensional linear subspace  $L$  of  $E$  the set  $\omega + L = \{\omega + x: x \in L\}$  is separable and metrizable it contains at most countably many elements of  $T$ . This shows that there exists  $I \subseteq T$  such that  $X := I \cup \{\omega\}$  is homeomorphic with  $T \cup \{\omega\}$  and  $\{x, y, \omega\}$  is affinely independent for  $x \neq y$  in  $I$  (choose a point from  $(\omega + L) \cap T$  whenever  $L$  is such that this set is nonempty and let  $I$  be the set of those points).

Now define a correspondence  $\Phi$  between  $X(3)$  and  $K$  by  $\Phi(A) := \text{conv } A$ . Then  $\Phi$  is l.s.c. (this follows from the definition of the Vietoris topology and from [6, Proposition 2.6]) and  $\Phi(A)$  is obviously closed and convex for all  $A$ , hence  $\Phi$  admits a continuous selection  $\phi$  since  $X(3)$  is compact (see e.g. [5]). But this means that there exists a continuous convex selection  $\phi: X(3) \rightarrow E$  in contradiction to our proposition. Hence  $K$  must be metrizable and the proof of Theorem 2 is complete.

**REMARK 2.** By Michael's theorem [6, Theorem 3.2"] a complete convex subset  $C$  of a locally convex space has property (S2) provided its natural uniformity is metrizable. It therefore seems to be a natural question now, whether (S2) actually characterizes metrizability for such a set, too. The answer is yes if  $C$  is locally compact and  $\sigma$ -compact (this follows from Theorem 2), and if  $C$  is a weak locally convex space (cf. [4, Satz 3.3]), but seems to be unknown in the general case.

**REMARK 3.** In connection with Theorem 2, the following questions seem to be of some interest:

(Q1) Is a compact convex set necessarily metrizable if each of its compact

convex subsets is an absolute extensor for compact spaces?

(Q2) Is a compact convex set necessarily metrizable if each of its compact convex subsets is dyadic? (This question was posed by the referee.)

It is clear from the preceding arguments that if the answer to the second question is yes, so is the answer to the first question; in addition, the proof of Theorem 2 could be considerably simplified if the answer to either of those questions were in the affirmative.

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