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TWO COUNTEREXAMPLES INVOLVING INNER FUNCTIONS

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ABSTRACT. Two questions about the existence of bounded analytic functions with prescribed behavior are answered using a construction technique of McLaughlin and Piranian. If A is a relatively closed, countable subset of the unit disc, the construction gives

(I) Function $f \in H^\infty$ so that the inner factor of $f - \alpha$ is a finite Blaschke product if and only if $\alpha \in A$.

(II) Inner function ϕ so that $\phi - \alpha$ has a discrete singular inner factor if and only if $\alpha \in A$.

A technique of McLaughlin and Piranian [2] is used to construct bounded analytic functions in the unit disc $\mathfrak{U}$ whose inner factors have prescribed properties. We refer to [3] for definitions of the various types of inner functions. If $f \in H^\infty(\mathfrak{U})$, then $\mathrm{Inn}(f)$ is used to denote the inner factor of f.

Let A be a relatively closed and countable subset of $\mathfrak{U}$. For convenience and without loss of generality, we assume $0 \in A$.

EXAMPLE I. *There exists a function* $f \in H^\infty(\mathfrak{U})$, $f(\mathfrak{U}) = \mathfrak{U}$, *so that* $\mathrm{Inn}(f - \alpha)$ *is a nonconstant finite Blaschke product if and only if* $\alpha \in A$.

EXAMPLE II. *There exists a discrete singular inner function* ϕ *so that* $\mathrm{Inn}(\phi - \alpha)$ *has a discrete singular inner factor if and only if* $\alpha \in A$.

Example I shows that a result of Carl Cowen [1, Corollary to Theorem 5] on commutants of analytic Toeplitz operators is strictly stronger than the earlier result of Thomson [4]. Specifically, if $A = \{0\}$, then the set

$$\{\beta \in \mathfrak{U}: \mathrm{Inn}(f - f(\beta)) \text{ is finite Blaschke}\}$$

is the finite set $f^{-1}(\{0\})$.

Example II answers negatively the final question in [3]. For $\alpha \in \mathfrak{U}$, let ψ_α be the Möbius transformation of $\mathfrak{U}$ given by

$$\psi_\alpha(z) = (z - \alpha)/(1 - \bar{\alpha}z), \qquad z \in \mathfrak{U}.$$

Let $A = \{0\} \cup \{1/n: n = 2, 3, \dots\}$. Then the inner function ϕ given by Example II is the limit in H^∞ of the sequence $\{(\psi_{1/n} \circ \phi): n = 2, 3, \dots\}$, and each of these has a nontrivial discrete singular inner factor.

Construction of Example I. Fix $\beta \in \mathfrak{U}$ so that the line segment $\Gamma = [\beta, 1)$ is contained in $\mathfrak{U} \setminus A$. Choose a sequence $\{\Omega_k: k = 0, 1, \dots\}$ of open

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subsets of $\mathfrak{U}$ so that for $k = 0, 1, 2, \ldots$ we have

(a) $\Gamma \subseteq \Omega_k \subseteq \mathfrak{U} \setminus A$,

(b) $\overline{\Omega}_k \cap \mathfrak{U} \subseteq \Omega_{k+1}$,

(c) Ω_k is connected, with Ω_0 not simply connected, and

(d) $\bigcup_{k=0}^{\infty} \Omega_k = \mathfrak{U} \setminus A$.

Denote by S_k the universal covering surface of Ω_k, $k = 0, 1, 2, \ldots$. Let $\gamma_0, \gamma_1, \ldots$ be a labelling of the segments on S_0 lying over Γ. For each $k = 1, 2, \ldots$, slit one sheet of S_k along a segment lying over Γ, slit S_0 along γ_k, and join S_k to S_0 by connecting opposite sides of these slits. Likewise, slit a copy of $\mathfrak{U}$ along Γ and join it to S_0 along γ_0. Let S denote the resulting simply connected Riemann surface with π the projection onto $\mathfrak{U}$. Let F be a conformal mapping of the unit disc onto S. We show that $f = \pi \circ F$ has the desired properties.

Clearly $f \in H^\infty$ and $f(\mathfrak{U}) = \mathfrak{U}$. If $\alpha \in \mathfrak{U} \setminus A$, then there is some k_0 with $\alpha \in \Omega_{k_0}$; hence $\alpha \in \Omega_k$ for all $k > k_0$, implying that $f - \alpha$ has infinitely many zeros. On the other hand, if $\alpha \in A$, then $f - \alpha$ has precisely one zero, and it remains only to show that $f - \alpha$ has no (nontrivial) singular inner factor.

Removing the sets $F^{-1}(\gamma_k)$, $k = 0, 1, 2, \ldots$ breaks $\mathfrak{U}$ into disjoint simply connected open sets $\Delta_0, \Delta_1, \ldots$, each Δ_k corresponding under F to the slit surface S_k, and Δ_{-1}, corresponding under F to the slit copy of $\mathfrak{U}$ which went into S. Suppose there exists $e^{i\theta} \in \partial\mathfrak{U}$ and r_0, $0 < r_0 < 1$, so that

$$|f(re^{i\theta}) - \alpha| < \delta, \qquad r_0 \leqslant r < 1, \tag{1}$$

where $\delta = \delta(\alpha) > 0$ is given by

$$\delta = \inf\{|\alpha - z| : z \in \Gamma \cup \partial\mathfrak{U}\}.$$

Then the segment $\{re^{i\theta} : r \geqslant r_0\}$ lies entirely within one of $\Delta_{-1}, \Delta_0, \Delta_1, \ldots$. It cannot lie in Δ_{-1} since the image $\{F(re^{i\theta}) : r \geqslant r_0\}$ would then be contained in a compact subset of S, contradicting the fact that F is a conformal mapping. Thus it lies in Δ_{k_0} for some $k_0 \geqslant 0$. But this implies that $\{f(re^{i\theta}) : r \geqslant r_0\} \subseteq \Omega_{k_0}$. The conditions imposed on the sets $\{\Omega_k\}$ imply $\alpha \notin \Omega_{k_0}$, so

$$\liminf_{r \to 1} |f(re^{i\theta}) - \alpha| > 0. \tag{2}$$

Since (1) implies (2) we conclude that $f - \alpha$ cannot have zero as a radial limit. As is well known, this implies $f - \alpha$ has no singular inner factor.

Construction of Example II. Fix a segment Γ as above. Let $\alpha_1, \alpha_2, \ldots$ be the nonzero points of A. (Of course, A may be finite, but that just makes our proof easier.) Let S_0 be the universal covering surface of $\mathfrak{U} \setminus \{0\}$ and S_k the universal covering surface of $\mathfrak{U} \setminus \{0, \alpha_k\}$, $k = 1, 2, \ldots$. Let $\gamma_1, \gamma_2, \ldots$ denote the segments on S_0 lying over Γ. As before, slit one sheet of each S_k along a segment lying over Γ, slit S_0 along γ_k, and join S_k to S_0 by connecting opposite sides of these slits. Let S denote the resulting simply connected Riemann surface, π the projection onto $\mathfrak{U} \setminus \{0\}$, and G a conformal mapping of the unit disc onto S. We show that $\phi = \pi \circ G$ has the desired properties.

First, all radial limits of ϕ lie in $A \cup \partial \mathfrak{U}$. A is countable, hence of capacity zero; so the set $\{e^{i\theta}: \lim_{r\to 1} \phi(re^{i\theta}) \in A\}$ is of Lebesgue measure zero. That is, almost all radial limits of ϕ are in $\partial \mathfrak{U}$, so ϕ is an inner function. Also, if $\alpha \in \mathfrak{U} \setminus A$, then α cannot be a radial limit of ϕ, implying that zero cannot be a radial limit of $\phi - \alpha$. Thus $\phi - \alpha$ has no singular inner factor. ϕ itself is singular since S does not cover $z = 0$.

Now fix $\alpha_k \in A$. As before, removing the sets $G^{-1}(\gamma_k)$ breaks $\mathfrak{U}$ into disjoint simply connected open sets $\Delta_0, \Delta_1, \ldots,$ each Δ_k corresponding under G to the slit surface S_k. α_k occurs as the asymptotic value on some arc in $\mathfrak{U}$. An argument similar to that used earlier shows that from some point on, such an arc lies entirely in Δ_k. Also, since α_k is a positive distance from Γ, such an arc must end at an interior point $e^{i\theta} \in \partial\Delta_k \cap \partial \mathfrak{U}$. We see then that $\phi - \alpha$ does not vanish in Δ_k, yet $\lim_{r\to 1}(\phi(re^{i\theta}) - \alpha) = 0$; so $\phi - \alpha$ has a singular inner factor. The counting argument used in [3] to prove Theorem I(a) can be applied here to show that the number of such points $e^{i\theta}$ in the interior of $\partial\Delta_k \cap \partial \mathfrak{U}$ is at most countable. Since these are the support points for the measure associated with the singular factor of ϕ, that factor must be a discrete singular inner function.

Finally, the same counting argument shows that $\{e^{i\theta}: \lim_{r\to 1} \phi(re^{i\theta}) = 0\}$ is at most countable, hence ϕ itself is a discrete singular inner function.

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