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EXISTENCE OF FIXED POINTS OF NONEXPANSIVE MAPPINGS IN CERTAIN BANACH LATTICES

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ABSTRACT. A fixed point theorem for nonexpansive mappings in dual Banach spaces is proved. Applications in certain Banach lattices are given.

1. Suppose K is a subset of a Banach space X and T: $K \to K$ is a nonexpansive mapping, i.e. $\|T(x) - T(y)\| \leqslant \|x - y\|$, $x, y \in K$. A well-known theorem due to Kirk [**1**] states that, if K is convex weakly compact (weak* compact when X is a dual space) and has normal structure, then T has a fixed point in K. In particular, if $X = L^p$ $(1 < p < \infty)$ Kirk's theorem applies to every bounded closed convex set K, while an analogous theorem was proved by Karlovitz [**3**, Corollary] in l^1. No result seems to be known in L^∞.

In this paper we study the existence of fixed points of nonexpansive mappings in certain (complex) AM-spaces. First we prove a general fixed point theorem for nonexpansive mappings in dual spaces and then we draw some consequences for nonexpansive mappings acting in (complex) AM-spaces which are dual to (complex) AL-spaces. These results imply, for instance, that every nonexpansive operator mapping into itself a closed ball $B \subseteq L^\infty$ has a fixed point in B, and every nonexpansive T: $L^\infty \to L^\infty$, which leaves invariant a weak* compact subset, has a fixed point (in L^∞).

2. A real Banach lattice X is called an AM-space (abstract-m-space) if $\|x \vee y\| = \|x\| \vee \|y\|$, for every $x, y \in X$ such that $x, y \geqslant 0$. Here and in the sequel $\vee$ and $\wedge$ denote the least upper bound and the greatest lower bound respectively. X is said to be order complete if each set $A \subseteq X$ with an upper bound has a least upper bound. A complex AM-space is defined as the complexification of an AM-space.

Suppose X is an order complete AM-space with unit (i.e. an element e such that the unit ball at zero is the order interval $[-e, e]$); then X is isometrically lattice isomorphic to the space $C_R(S)$ of all continuous real-valued functions defined on a compact Stonian space S.

For these and other facts about Banach lattices we refer to Schaefer's book [**4**].

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The following notation will be used throughout the paper: for a Banach space X, $B(x, r)$ denotes the closed ball centered at $x \in X$ of radius r; if $M \subseteq X$ is a nonvoid bounded subset, diam M denotes the diameter of M and $\overline{\text{co}}\, M$ the closed convex hull of M.

LEMMA. *Suppose X is the complexification of an order complete AM-space with unit. For every nonvoid bounded closed set $M \subseteq X$ there exists a point $z_M \in X$ with the following properties*:

(a) $M \subseteq B(z_M, 2^{-1/2} \operatorname{diam} M)$.

(b) *If, for some $y \in X$, $M \subseteq B(y, 2^{-1/2} \operatorname{diam} M)$, then $\|z_M - y\| \leqslant 2^{-1/2}$* diam M.

A similar statement holds if X is an order complete AM-space with unit, with $2^{-1/2}$ replaced by $1/2$.

PROOF. We shall prove the lemma in the complex case, the real one being simpler. Let $C(S)$ denote the space of continuous complex-valued functions which represents X. Set:

$$a_1 = \bigvee_{f \in M} \operatorname{Re} f, \quad a_2 = \bigwedge_{f \in M} \operatorname{Re} f, \quad a_3 = \bigvee_{f \in M} \operatorname{Im} f, \quad a_4 = \bigwedge_{f \in M} \operatorname{Im} f.$$

By our assumption on X the a_j's are continuous real-valued functions belonging to $C_R(S)$. Let $u = (a_1 + a_2)/2$, $v = (a_3 + a_4)/2$. We shall prove that $z_M = u + iv$ has the desired properties. Indeed, if $r = \operatorname{diam} M$, it is easy to see that $\|a_1 - a_2\| \leqslant r$ and $\|a_3 - a_4\| \leqslant r$ so that, for every $s \in S$ and $f \in M$, $|\operatorname{Re} f(s) - u(s)| \leqslant r/2$, $|\operatorname{Im} f(s) - v(s)| \leqslant r/2$; hence (a) holds.

Suppose now that $M \subseteq B(y, 2^{-1/2} \operatorname{diam} M)$ for some $y \in X$. Let $\varepsilon > 0$ be arbitrarily small. For every $s \in S$ we can find a neighborhood of s, $V(s)$, such that $|a_j(t_1) - a_j(t_2)| < \varepsilon$ ($j = 1, 2, 3, 4$) and $|y(t_1) - y(t_2)| < \varepsilon$ whenever $t_1, t_2 \in V(s)$.

Since S is extremally disconnected, we have:

$$a_1(s) = \inf_{U(s)} \sup_{t \in U(s)} \sup_{f \in M} \operatorname{Re} f(t)$$

where $U(s)$ runs through a neighborhood base of s (see [4, p. 107]). Hence there is a point $s_1 \in V(s)$ and a function $f_{1,s} \in M$ such that: $|\operatorname{Re} f_{1,s}(s_1) - a_1(s)| < \varepsilon$. Therefore

$$|a_1(s) + i \operatorname{Im} f_{1,s}(s_1) - y(s)| < 2\varepsilon + 2^{-1/2} r.$$

Analogously, we can find points $s_j \in V(s)$ and functions $f_{j,s} \in M$ such that $|g_j(s) - y(s)| < 2\varepsilon + 2^{-1/2} r$, where

$$g_j(s) = a_j(s) + i \operatorname{Im} f_{j,s}(s_j) \quad \text{if } j = 1, 2,$$
$$g_j(s) = \operatorname{Re} f_{j,s}(s_j) + i a_j(s) \quad \text{if } j = 3, 4.$$

Since the oscillation of the a_j's on $V(s)$ is less than ε, an elementary geometric argument shows that there is a number in the convex hull of the $g_j(s)$'s whose distance from $z_M(s)$ is less than ε. Therefore $|z_M(s) - y(s)| < 3\varepsilon + 2^{-1/2} r$ for all $s \in S$, whence (b).

REMARK. It follows from the above construction that if B is a closed ball containing M, then $z_M \in B$. Analogously, if M is contained in some order interval $I = \{x \in X: a \leqslant x \leqslant b\}$, then $z_M \in I$ (in the case X is an AM-space). Therefore we are led to the following definition.

DEFINITION. A closed subset K of a Banach space X has uniform relative normal structure if there exists $c < 1$ such that, for every nonvoid bounded closed subset $M \subseteq K$, there exists $z_M \in K$ with the following properties:

(a′) $M \subseteq B(z_M, c \operatorname{diam} M)$.

(b′) If, for some $y \in K$, $M \subseteq B(y, c \operatorname{diam} M)$, then $\|z_M - y\| \leqslant c \operatorname{diam} M$.

This definition should be compared with the analogous definition in **[2]**.

THEOREM. *Suppose X is a dual Banach space and $K \subseteq X$ a weak* closed set with uniform relative normal structure. Let T: $K \to K$ be a nonexpansive mapping which leaves invariant a weak* compact subset $M \subseteq K$ (i.e. $T(M) \subseteq M$). Then there exists $u \in K$ such that $u = T(u)$.*

PROOF. Let $A_0 \subseteq M$ be minimal among weak* compact invariant subsets of M. Then, if cl* denotes the weak* closure,

$$T(\mathrm{cl}^* T(A_0)) \subseteq T(A_0) \subseteq \mathrm{cl}^* T(A_0)$$

so that $A_0 = \mathrm{cl}^* T(A_0)$. Suppose r is the diameter of A_0. Then the set $A = \{z \in K: A_0 \subseteq B(z, cr)\}$ is nonvoid, since $z_{A_0} \in A$. Moreover A is weak* compact, as an intersection of closed balls with K. Fix $\varepsilon > 0$ arbitrarily. For every $z \in A$ and $x \in A_0$ there exists $y \in A_0$ such that:

$$\|T(z) - x\| - \varepsilon \leqslant \|T(z) - T(y)\| \leqslant \|z - y\| \leqslant cr.$$

Since ε is arbitrary, $\|T(z) - x\| \leqslant cr$ and $T(A) \subseteq A$.

Define a set H by $H = \{z \in A: A \subseteq B(z, cr)\}$. H is nonvoid since $z_{A_0} \in H$. Let A_1 denote the intersection of all w^* compact invariant subsets of A containing H. An argument due essentially to Kirk **[1]** shows that $\operatorname{diam} A_1 \leqslant cr$. Namely, let F denote the set $\{z \in A_1: A_1 \subseteq B(z, cr)\}$. F contains H and is weak* compact (as an intersection of closed balls with A_1). Assume, by way of contradiction, that $T(z) \notin F$ for some $z \in F$; then the set $G = B(T(z), cr) \cap A_1$ is weak* compact and contains H. Moreover, for every $x \in G$: $\|T(z) - T(x)\| \leqslant \|z - x\| \leqslant cr$, since $z \in F$. Hence $T(G) \subseteq G$ and, by the definition of A_1, $A_1 = G$. But $T(z) \notin F$, so that $\|T(z) - x\| > cr$ for some $x \in A_1$, a contradiction. Consequently $T(F) \subseteq F$, and by the definition of A_1 again, $F = A_1$. Hence $\operatorname{diam} A_1 = \operatorname{diam} F \leqslant cr$. Moreover $A_0 \subseteq B(x, cr)$ for every $x \in A_1$. Repeating this construction, we define inductively a sequence of weak* compact subsets $A_n \subseteq K$ with the properties:

(i) $\operatorname{diam} A_n \leqslant rc^n$,

(ii) $T(A_n) \subseteq A_n$,

(iii) $\|x - y\| \leqslant rc^n$, whenever $x \in A_n, y \in A_{n-1}$.

If we pick a sequence of points $u_n \in A_n$ we have:

$$\|u_n - u_m\| \leqslant r\sum_n^m c^j \quad (n < m) \quad \text{and} \quad \|u_n - T(u_n)\| \leqslant rc^n.$$

Therefore u_n converges in the norm topology to a point $u \in K$ such that $u = T(u)$.

We recall that an AL-space is a real Banach lattice such that $\|x + y\| = \|x\| + \|y\|$ whenever $x, y \geqslant 0$. A complex AL-space is defined to be the complexification of a real AL-space. It is known **[4]** that the dual of a (complex) AL-space is a (complex) AM-space which satisfies the assumptions of the above lemma. Henceforth we have the following consequences.

COROLLARY 1. *Suppose X is the dual of a (complex) AL-space. Then, if $B \subseteq X$ is a closed ball and T: $B \to B$ is a nonexpansive mapping, T has a fixed point in B.*

COROLLARY 2. *Suppose X is the dual of an AL-space. If $I \subseteq X$ is a closed order interval and T: $I \to I$ is a nonexpansive mapping, T has a fixed point in I.*

COROLLARY 3. *Suppose X is the dual of a (complex) AL-space. If T: $X \to X$ is a nonexpansive mapping which leaves invariant a weak* compact subset of X, T has a fixed point (in X).*

REMARK. Suppose (Y, Σ, μ) is a σ-finite measure space; the dual of the (complex) AL-space $L^1(Y, \Sigma, \mu)$ is identified with $L^\infty(Y, \Sigma, \mu)$, so that the above corollaries hold with $L^\infty(Y, \Sigma, \mu)$ in place of X.

3. In this section we give an example of uniform relative normal structure in spaces which are not AM-spaces.

Suppose X is a uniformly convex Banach space; denote by X^* its dual space and by $\|\cdot\|$ and $\|\cdot\|^*$ the norms in X and X^* respectively. Let Z denote the space of all sequences $z = (z_1, z_2, \ldots, z_n, \ldots)$, $z_n \in X$, such that $\sup_n\|z_n\| = \|z\|_\infty < \infty$. Z is not an AM-space unless X itself is an AM-space. Moreover Z is the dual of the space of all sequences $t = (t_1, t_2, \ldots, t_n, \ldots)$, $t_n \in X^*$, such that $\Sigma_n\|t_n\|^* < \infty$.

PROPOSITION. *Z and every closed ball $B \subseteq Z$ have uniform relative normal structure.*

PROOF. The proof is achieved by generalizing the argument used in the lemma of §2. Suppose $C \subseteq Z$ is a closed nonvoid bounded set. Let $z = (z_1, z_2, \ldots, z_n, \ldots)$ belong to C and denote by C_n the subset of X described by z_n as z describes C. Since X is uniformly convex, there exists $c < 1$, independent from C and n, such that there exist points $z_{C,n} \in \overline{\text{co}}\, C_n$ with the property $\|z_{C,n} - z_n\| \leqslant cr$ for every $z_n \in C_n$, $n = 0, 1, 2, \ldots,$ (here we made $r = \text{diam}\, C$). Thus the point $z_C = (z_{C,1}, \ldots, z_{C,n}, \ldots)$ has the property (a′). On the other hand suppose that the point $y = (y_1, y_2, \ldots, y_n, \ldots)$ is such that $C \subseteq B(y, cr)$. It follows that $C_n \subseteq B(y_n, cr)$ for every n. By the definition of $z_{C,n}$, we have $z_{C,n} \in B(y_n, cr)$ too, so that $\|z_C - y\|_\infty \leqslant cr$. It is also clear that $z_C \in B$ if C is contained in the closed ball B.

From this proposition it is possible to deduce the analogues of Corollaries 1 and 3.

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