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ON A CONJECTURE OF HUNT AND MURRAY CONCERNING q-PLURISUBHARMONIC FUNCTIONS

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ABSTRACT. We discuss the conjecture of Hunt and Murray on uniqueness of the Dirichlet problem for the generalized complex Monge-Ampère equation. We define a class of q-plurisubharmonic functions, prove uniqueness in this class, and show that in some cases the solution found by Hunt and Murray is in our class.

Introduction. In [4] Hunt and Murray generalize the Perron-Bremermann method [3] of solving the Dirichlet problem for generalized complex Monge-Ampère equation to a class of q-plurisubharmonic functions, P_q. They conjecture that their extremal function, $\bar{u}(z) = \sup\{v(z)\colon v \in P_q, v \leqslant b$ on $\partial\Omega\}$, is the unique solution of the problem, $u \in P_q(\Omega) \cap (-P_{n-q-1}(\Omega))$, $u \in C(\bar{\Omega})$, $u|\partial\Omega = b(z)$, in the class P_q. What we do here is define a somewhat smaller class $\tilde{P}_q$. We show that for this class the Dirichlet problem above indeed has a unique solution. We show further that in certain special cases $\bar{u}(z)$ actually solves the Dirichlet problem for $\tilde{P}_q$ and hence is the unique solution, in $\tilde{P}_q$. In our discussion we need to assume that $2q < n$ and $\partial\Omega$ is strictly q-pseudoconvex-exactly as in [4].

1. $\tilde{P}_q(\Omega)$. Let $\Omega \subseteq \mathbf{C}^n$ be a domain. We begin by recalling the definition of q-plurisubharmonic function as found in [4].

(1.1) DEFINITION. (A) $u\colon \Omega \to \mathbf{R} \cup \{-\infty\}$ is said to be $(n-1)$ plurisubharmonic if

(1) u is upper semicontinuous on Ω.

(2) If $B \subset \Omega$ is a ball and g is a lower semicontinuous plurisuperharmonic function on B, then $g \geqslant u$ on $\partial B \Rightarrow g \geqslant u$ on B.

(B) u is said to be q-plurisubharmonic if u is q-plurisubharmonic on $\Omega \cap \Pi_{q+1}$, where Π_{q+1} is a complex linear $(q+1)$ dimensional subspace of $\mathbf{C}^n$, with $\Pi_{q+1} \cap \Omega \neq \varnothing$.

We will use the notation $P_q(\Omega)$ for the class of all q-plurisubharmonic functions on Ω, and $PS_q(\Omega) = -P_q(\Omega)$ for the class of q-plurisuperharmonic functions. We also note that since plurisuperharmonic functions are the increasing limit of smooth plurisuperharmonic functions, it is equivalent to define $P_q(\Omega)$ as functions (upper semicontinuous) which satisfy a maximum

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principle with respect to smooth plurisuperharmonic functions.

We now define another class of q-plurisubharmonic functions, contained in $P_q(\Omega)$. It is for this class of functions that we are able to prove uniqueness in the Dirichlet problem.

(1.2) DEFINITION. (A) Let $\tilde{P}_q(\Omega)$ denote the class of functions which are upper semicontinuous on Ω and which satisfy the following condition: if $B \subset \Omega$ is a ball and $g \in PS_{n-q-1}(\Omega)$ satisfies $g \geqslant u$ on ∂B, then $g \geqslant u$ on B.

(B) We define $\widetilde{PS}_q(\Omega) = -\tilde{P}_q(\Omega)$ as functions satisfying a minimum principle with respect to $P_{n-q-1}(\Omega)$.

(1.3) PROPOSITION. $\tilde{P}_q(\Omega) \subset P_q(\Omega)$.

PROOF. The idea here is the same as the first part of the proof of Theorem 3.3 of [4]. Let $u \in \tilde{P}_q(\Omega)$ and let Π_{q+1} be a $(q+1)$ dimensional complex plane intersecting Ω. After a linear change of coordinates we can assume that $\Pi_{q+1} = \{z^1 = \cdots = z^{n-q-1} = 0\}$. Let $g \in PS_0(\Pi_{q+1})$ be such that $g \geqslant u$ on $\partial B \cap \Pi_{q+1}$, where $B \subset \Omega$ is a ball. We extend g to B by $\tilde{g} = f(\sum_{j=1}^{n-q-1}|z^j|^2) + g(z^{n-q}, \ldots, z^n)$. It is easy to see that $\tilde{g}$ is in $PS_{n-q-1}(\Omega)$. We choose f so that f is lower semicontinuous, $f(0) = 0$ and $\tilde{g} \geqslant u$ on ∂B. Then $\tilde{g} \geqslant u$ on $B \Rightarrow g \geqslant u$ on $B \cap \Pi_{q+1}$.

(1.4) REMARKS. (1) It is easy to see that $\tilde{P}_0(\Omega) = P_0(\Omega)$ is just the class of plurisubharmonic functions, and that $\tilde{P}_{n-1}(\Omega) = P_{n-1}(\Omega)$.

(2) It seems unlikely to us that $\tilde{P}_q(\Omega) = P_q(\Omega)$ for $0 < q < n-1$.

Even if we define $\tilde{P}_q(\Omega)$ as those functions satisfying a maximum principle with respect to smooth $(n-q-1)$ plurisuperharmonic functions, it seems that one would have to know that such functions are plurisuperharmonic functions on some $(q+1)$ dimensional local complex analytic submanifold through every point. This is not true, for example if the complex Hessian does not have constant rank and has at least $(q+1)$ eigenvalues equal to zero at every point (for a discussion of this see [2]). Even if the complex Hessian does have constant rank but there are some positive eigenvalues Eric Bedford has communicated a counterexample to us, based on the function given on the bottom half of p. 4.11 of [1].

2. The generalized Dirichlet problem.

(2.1) DEFINITION. We say that $u \in \tilde{P}_q(\Omega)$ satisfies the generalized complex Monge-Ampère equation in $\tilde{P}_q$ if $u \in \tilde{P}_q(\Omega) \cap \widetilde{PS}_{n-q-1}(\Omega)$.

We note that if u satisfies the generalized complex Monge-Ampère equation then u, being both upper and lower semicontinuous, is continuous. Now if $u \in C^2(\Omega) \cap \tilde{P}_q(\Omega) \cap \widetilde{PS}_{n-q-1}(\Omega)$ then in particular $u \in C^2(\Omega) \cap P_q(\Omega) \cap PS_{n-q-1}(\Omega)$. In [4] it is shown that $C^2(\Omega) \cap P_q(\Omega) = \{u \in C^2(\Omega)$: $dd^cu(p)$ has at least $(n-q)$ nonnegative eigenvalues for all $p\}$, and $C^2(\Omega) \cap PS_{n-q-1}(\Omega) = \{u \in C^2(\Omega)$: $dd^cu(p)$ has at least $(q+1)$ nonpositive eigenvalues for all $p\}$. So if u satisfies the generalized Monge-Ampère equation in

$\tilde{P}_q$ and is in $C^2(\Omega)$, then $dd^cu(p)$ must have at least one zero eigenvalue, so

$$\underbrace{dd^cu \wedge \cdots \wedge dd^cu}_{n\text{-times}} = 0.$$

We cannot characterize $C^2(\Omega) \cap \tilde{P}_q$ quite so simply as was done in **[4]**. However we can prove the following result.

(2.2) PROPOSITION. *Let* $\mathcal{H}^+_{n-q} = \{u \in C^2(\Omega)$: $dd^cu(p)$ *has at least* $(n-q)$ *positive eigenvalues at each point* $p \in \Omega\}$. *Then we have* $\mathcal{H}^+_{n-q} \subset \tilde{P}_q(\Omega)$.

PROOF. To prove this result all we need note is that we have only to prove the result locally. Now, working at a fixed point $0 \in \Omega$, we can assume that $dd^cu(0)$ is positive definite on the plane $\Pi = \{z^{n-q-1} = \cdots = z^n = 0\}$. Then in a neighborhood of 0, u is plurisubharmonic on Π. Then if $B \subset \Pi \cap \Omega$ is a ball and $g \in PS_{n-q-1}(\Pi)$ is such that $g \geqslant u$ on ∂B we have that $g \geqslant u$ on B.

In order to prove uniqueness of the Dirichlet problem we will need the following result:

(2.3) PROPOSITION. $\tilde{P}_{q_1}(\Omega) + \tilde{P}_{q_2}(\Omega) = \tilde{P}_{n-1}(\Omega) = P_{n-1}(\Omega)$ *if* $q_1 + q_2 = n - 1$.

Before we prove this we give the following lemma needed in the proof.

(2.4) LEMMA. $\tilde{P}_0(\Omega) + \tilde{P}_q(\Omega) \subset \tilde{P}_q(\Omega)$.

PROOF. Suppose $g \in \tilde{P}_0$, $u \in \tilde{P}_q$ and $h \in PS_{n-q-1}$ are such that $h \geqslant g + u$ on ∂B where B is a ball in Ω. Then it suffices to show that $h - g \in PS_{n-q-1}$. Let Π_{n-q} be an $(n-q)$ complex dimensional plane such that $\Pi_{n-q} \cap \Omega \neq \varnothing$. Then let $k \in P_0(\Pi_{n-q})$, $k \leqslant h - g$ on $\partial B'$ where B' is a ball in $\Pi_{n-q} \cap \Omega$. Then $k + g \in P_0(\Pi_{n-q})$ so $k + g \leqslant h$ in B'.

Now we are able to give the proof of Proposition 2.3.

PROOF. Let $u_i \in P_{q_i}(\Omega)$, $i = 1, 2$. Let $g \in PS_0$ be such that $g \geqslant u_1 + u_2$ on ∂B; where, again, $B \subset \Omega$ is a ball. Then $g - u_2 \in PS_{n-q-1}$ by Lemma 2.4 and $g - u_2 \geqslant u_1$ on ∂B. So the definition of $\tilde{P}_{q_2}$ implies that $u_1 \leqslant g - u_2$ on B.

(2.5) THEOREM. *Suppose* $u_1, u_2 \in C(\Omega) \cap \tilde{P}_q(\Omega)$ *are both solutions of the generalized complex Monge-Ampère equation in* $P_q(\Omega)$, *and further suppose* $u_1|\partial\Omega = u_2|\partial\Omega$. *Then* $u_1 = u_2$ *in* Ω.

PROOF. $u_1 \in \tilde{P}_q(\Omega) \cap \widetilde{PS}_{n-q-1}(\Omega)$. We will use Proposition 2.3. $u_1 - u_2 \in \tilde{P}_q(\Omega) + \tilde{P}_{n-q-1}(\Omega) \subset \tilde{P}_{n-1}(\Omega)$ since $u_2 \in \widetilde{PS}_{n-q-1}(\Omega)$. This gives that $u_1 - u_2 \leqslant 0$ in Ω. But on the other hand $u_2 - u_1 \in \tilde{P}_q(\Omega) - \tilde{P}_{n-q-1}(\Omega)$ since $u_1 \in PS_{n-q-1}(\Omega)$. So we have $u_2 - u_1 \leqslant 0$ on Ω.

Let $\Omega \subset \mathbf{C}^n$ be a strictly q-pseudoconvex domain with smooth boundary. We consider the following generalized Dirichlet problem for the complex

Monge-Ampère equation. For $b(z) \in C(\partial\Omega)$, let

$$\mathfrak{B}_q(b, \Omega) = \{v \in P_q(\Omega): u(z) \leqslant b(z) \text{ on } \partial\Omega\}.$$

Let $u(z) = \sup_{v \in \mathfrak{B}_q(b,\Omega)} v(z)$. Then if $2q < n$ it is shown in [4] that

(1) $\bar{u}(z)|\partial\Omega = b(z)$.

(2) $\bar{u}(z) \in C(\Omega)$.

(3) $\bar{u}(z) \in P_q(\Omega) \cap PS_{n-q-1}(\Omega)$.

Now our uniqueness Theorem 2.5 holds only if $\bar{u} \in \tilde{P}_q(\Omega) \cap \widetilde{PS}_{n-q-1}(\Omega)$. We show now that in two (very special) cases, this is indeed the case.

(2.6) THEOREM. *Let $\Omega \subseteq \mathbf{C}^{2n+1}$ be a strictly n-pseudoconvex with smooth boundary. If $\bar{u}(z) = \sup_{v \in \mathfrak{B}_n(b,\Omega)} v(z)$, we have that $\bar{u}(z)$ satisfies the generalized complex Monge-Ampère equation in $\tilde{P}_n(\Omega)$, i.e. $\bar{u} \in \tilde{P}_n(\Omega) \cap \widetilde{PS}_n(\Omega)$.*

PROOF. If we examine the proof of Theorem 3.3 of [4] we note that what is actually being proved there is that $\bar{u}(z) \in \widetilde{PS}_{m-q-1}(\Omega)$ for $\Omega \subset \mathbf{C}^m$, Ω strictly q-pseudoconvex in $\mathbf{C}^m$ with smooth boundary. Now we consider the lower envelope problem dual to the one defining $\bar{u}(z)$. Let $\mathfrak{B}^S_{m-q-1}(b, \Omega)$ denote the class of functions in PS_{m-q-1} which are greater than or equal to $b(z)$ on $\partial\Omega$. Then if Ω is strictly $(m - q - 1)$ pseudoconvex and $2(m - q - 1) < m$ we get a function $\bar{u}'$ defined by

$$\bar{u}'(z) = \inf_{v \in \mathfrak{B}^S_{m-q-1}(b,\,\Omega)} v(z).$$

Using the arguments of [3], [4], it is easy to see that $\bar{u}'(z)$ satisfies

(1) $\bar{u}'(z) \in PS_{m-q-1}(\Omega) \cap \tilde{P}_q$.

(2) $\bar{u}' \in C(\Omega)$.

(3) $\bar{u}'|\partial\Omega = b$.

Now in the case where $m = 2n + 1$, $q = n$, and Ω is strictly n-pseudoconvex with smooth boundary both hypotheses (for $\mathfrak{B}_q$ and for $\mathfrak{B}^S_{n-q-1}$) are satisfied and we get two functions $\bar{u}(z)$ and $\bar{u}'(z)$. We claim that $\bar{u}(z) = \bar{u}'(z)$. To see this, firstly we note that $\bar{u}' \geqslant \bar{u}$ on $\partial\Omega$. So since $\bar{u}' \in \widetilde{PS}_{m-q-1}(\Omega)$ and $\bar{u} \in P_q(\Omega)$ we get $\bar{u}' \geqslant \bar{u}$ in Ω. Similarly $\bar{u}' \leqslant \bar{u}$ on $\partial\Omega$, and $\bar{u} \in \tilde{P}_q$ and $\bar{u} \in PS_{n-q-1}$ so $\bar{u}' \leqslant \bar{u}$ in Ω. So $\bar{u} = \bar{u}'$ and $\bar{u} \in \tilde{P}_n(\Omega) \cap \widetilde{PS}_n(\Omega)$.

Now if Ω is strictly q-pseudoconvex with smooth boundary we can form the class

$$\tilde{\mathfrak{B}}_q(b, \Omega) = \{u \in \tilde{P}_q(\Omega): u(z) \leqslant b(z) \text{ on } \partial\Omega\}.$$

Then as in Bremermann [3], we can form the upper envelope $\tilde{u}(z)$ defined by $\tilde{u}(z) = \sup_{v \in \tilde{\mathfrak{B}}_q(b,\Omega)} v(z)$ and ask whether $\tilde{u}(z)$ satisfies the generalized complex Monge-Ampère equation for $\tilde{P}_q(\Omega)$. We do not know the answer to this in general but in case $\bar{u}(z)$ satisfies the complex Monge-Ampère equation in $\tilde{P}_q(\Omega)$ we get the following result.

(2.7) PROPOSITION. *If $\bar{u}$ satisfies the complex Monge-Ampère equation for $\tilde{P}_q(\Omega)$, then $\bar{u}(z) = u(z)$.*

PROOF. Since $P_q(\Omega) \supset \tilde{P}_q(\Omega)$, it is clear that $\tilde{u}(z) \leqslant \bar{u}(z)$. Now since $\bar{u} \in \tilde{P}_q(\Omega)$ we get that $\bar{u} \in \tilde{\mathfrak{B}}_q(b, \Omega)$, so $\tilde{u}(z) = \sup_{v \in \tilde{\mathfrak{B}}_q(b, \Omega)} v(z) \geqslant \bar{u}(z)$.

There is another instance when we can conclude that $\bar{u}$ satisfies that complex Monge-Ampère equation for $\tilde{P}_q(\Omega)$. Suppose Ω is strictly pseudoconvex with smooth boundary. Then if we examine the proof of Theorem 3.2 of [**4**], we note that in this case it is not necessary to hypothesize that $2q < n$. Thus we get the following result.

(2.8) PROPOSITION. *If Ω is strongly pseudoconvex with smooth boundary then $\bar{u}$, the upper envelope of elements in $P_q(\Omega)$ which are $\leqslant b(z)$ on $\partial\Omega$, is in $\tilde{P}_q(\Omega) \cap \widetilde{PS}_{n-q-1}(\Omega)$, and is the unique solution of the generalized complex Monge-Ampère equation in $\tilde{P}_q(\Omega)$, and is equal to $\tilde{u}_1$ the upper envelope of elements of $\tilde{P}_q(\Omega)$ which are $\leqslant b(z)$ on $\partial\Omega$.*

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