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FIXED POINT THEOREMS FOR MULTIVALUED APPROXIMABLE MAPPINGS

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ABSTRACT. In this paper we introduce several classes of multivalued approximable mappings and develop the fixed point theory for these mappings acting in a cone. As an important special case we have the theory of k-ball-contractive perturbations of strongly pseudo-contractive and accretive mappings.

1. Introduction. Since the early 1960's several important classes of nonlinear noncompact mappings have been introduced among which are monotone type, condensing and A-proper mappings. The first two classes of mappings have been extensively studied also in the multivalued case by many authors, while the class of multivalued A-proper mappings has been first extensively studied in the author's thesis (1975) and his subsequent work and then jointly by the author and Petryshyn (see, e.g., [7], [8], [9]). It turns out that multivalued mappings play an important role in the theory of evolution equations, variational inequalities, contingent ordinary, partial differential and integral equations, optimal control theory, etc.

The purpose of this paper is to introduce and study several new classes of multivalued approximable mappings acting in a cone. A portion of these results pertaining to P_γ-compact mappings have been obtained in the author's Ph.D. thesis [7]. All our results are of the constructive nature as defined below. Applications of these results to boundary value problems for differential equations will be given in a subsequent paper.

2. Fixed point theorems for approximable mappings. We begin this section by introducing the classes of mappings to be studied in the sequel. Let X be a normed linear space and $\{X_n\}$ a sequence of oriented finite dimensional Banach spaces such that $\dim X_n \to \infty$ as $n \to \infty$. Suppose that for each n there exists a linear mapping V_n from X_n into X such that for some constant $C_1 > 0$, $\|V_n x\| \leqslant C_1\|x\|$ for all x in X_n and all n. The pair $\{X_n, V_n\}$ is referred to as *an approximation scheme* for X.

DEFINITION 1. If $X_n \subset X$, $P_n: X \to X_n$ is a continuous linear projection onto X_n such that $P_n x \to x$ in X as $n \to \infty$ for each x in X and V_n is the identity

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injection of X_n into X, then the triple of sequences $\Gamma = \{X_n, V_n, P_n\}$ is said to be a *projectionally complete scheme* for X.

Such a scheme always exists if X has a Schauder basis. Our first class of multivalued mappings is given by

DEFINITION 2. A multivalued mapping $T: D \subset X \to 2^X$ is said to be *approximately compact* (*A_γ-compact* for short) with respect to the scheme $\{X_n, V_n\}$ if

(1) there exists a sequence of upper semicontinuous mappings $\{T_n\}$ from $D_n \subset X_n$ into $CK(X_n)$ = the family of all nonempty compact and convex subsets of X_n;

(2) there exists $\gamma \geqslant 0$ such that, if for some positive $\mu \geqslant \gamma$ and a sequence $\{x_{n_k} | x_{n_k} \in D_{n_k}\}$ with $\|x_{n_k}\| \leqslant M$ for all k and some $M > 0$, we have that $V_{n_k}(y_{n_k} - \mu x_{n_k}) \to f$ as $k \to \infty$ in X for some $y_{n_k} \in T_{n_k}(x_{n_k})$ and f in X, then there exist a subsequence $\{x_{n_{k(i)}}\}$ and $x_0 \in X$ such that $V_{n_{k(i)}}(x_{n_{k(i)}}) \to x_0$ in X and $f \in T(x_0) - \mu x_0$.

When Definition 2 holds with f given in advance, we say that T is *A_γ-compact at f.*

DEFINITION 3. We say that $T: D \subset X \to 2^X$ is *projectionally compact* (P_γ-compact for short) w.r.t. a projectionally complete scheme $\Gamma = \{X_n, V_n, P_n\}$ if T, $T_n = P_n T$ and $D_n = D \cap X_n$ satisfy all the conditions in Definition 2. Moreover, if condition (2) holds only for $\mu = 1$, $T - I$ is said to be A-proper w.r.t. Γ.

REMARK. P_γ-compact single-valued mappings were introduced by Petryshyn [**10**] and by Milojević [**7**] in the multivalued case. A slight variant of the definition of A_1-compact mappings at 0 in the single-valued case was given in Lees-Schultz [**6**].

In [**7**] we have shown that if $T: D \subset X \to 2^X$ is either ball-condensing or k-contractive, generalized contractive or monotone, then T is P_1-compact. Moreover, perturbations of P_γ-compact mappings by compact multivalued ones are also P_γ-compact. We shall also need the following special case of Proposition 2.1 in [**8**].

EXAMPLE 1 [**8**]. Let X be a Banach space with a projectionally complete scheme $\Gamma = \{X_n, V_n, P_n\}$, $\|P_n\| \leqslant 1$, $T: X \to X$ continuous, surjective and a-stable, i.e. for some $c > 0$, $\|P_n Tx - P_n Ty\| \geqslant c\|x - y\|$ for all x, y in X_n and $n \geqslant 1$ and $F: D \subset X \to X$ demicontinuous and k-ball-contractive [**17**] with $k < c$ or ball-condensing if $c = 1$. Then $T + F$ is A-proper. In particular, as T we can take a c-strongly accretive map, i.e. $(Tx - Ty, x - y)_+ = \sup\{(Tx - Ty, u) | u \in J(x - y)\} \geqslant c\|x - y\|^2$, $x, y \in X$, where J is the normalized duality map.

By a cone $K \subset X$ we mean a closed subset of X such that $\alpha x + \beta y \in K$ whenever $x, y \in K$ and $\alpha \geqslant 0$, $\beta \geqslant 0$. We denote by $B(0, r)$ and $B_n(0, r)$ the open balls in X and X_n respectively centered at the origin and of radius r; $\partial_K(B \cap K)$ denotes the boundary of B relative to K.

Our first fixed point result for positive A_γ-compact mappings is given by

THEOREM 1. *Let X be a Banach space, K and K_n cones in X and X_n respectively, $D \subset X$ and $D_n \subset X_n$ open and bounded subsets containing the origin of X and X_n respectively and $T\colon \bar{D} \cap K \to 2^K$ A_γ-compact at 0 with $T_n\colon \bar{D}_n \cap K_n \to CK(K_n)$. Suppose that $\{D_n\}$ is uniformly bounded, i.e., there exists an $M > 0$ such that $\|x\| \leqslant M$ for all $x \in D_n$ and all n, and that T_n satisfies the Leray-Schauder condition on $\partial_{K_n}(D_n \cap K_n)$, i.e., for all large n*

(μ) *$\alpha x \notin T_n(x)$ for all $x \in \partial_{K_n}(D_n \cap K_n)$ and $\alpha \geqslant \mu$ for some positive $\mu \geqslant \gamma$.*

Then the equation $\mu x \in T(x)$ is feebly approximation solvable, i.e., for sufficiently large n, each equation $\mu x \in T_n(x)$ has a solution $x_n \in D_n \cap K_n$ and there exists a subsequence $\{x_{n_k}\}$ such that $V_{n_k}(x_{n_k}) \to x_0$ with $\mu x_0 \in T(x_0)$. If the equation $\mu x \in T(x)$ is uniquely solvable, then it is strongly approximation solvable in the sense that the entire sequence $V_n(x_n) \to x_0$ with $\mu x_0 \in T(x_0)$.

PROOF. For a large n define a mapping $H_n\colon [0, 1] \times (\bar{D}_n \cap K_n) \to CK(K_n)$ by $H_n(t, x) = t/\mu T_n(x)$. It is easy to see that H_n is u.s.c. compact and $x \notin H_n(t, x)$ for all $x \in \partial_{K_n}(D_n \cap K_n)$ and $t \in [0, 1]$ and by the homotopy theorem for the generalized fixed point index [2] $i_{K_n}(1/\mu T_n, D_n) = i_{K_n}(\hat{0}, D_n) = 1$, where $\hat{0}$ is the zero mapping. Hence, for all large n there exists $x_n \in D_n \cap K_n$ such that $\mu x_n \in T_n(x_n)$. Since $\|x_n\| \leqslant M$ for all n and $V_n(y_n - \mu x_n) = 0$ for $y_n \in T_n(x_n)$ with $\mu x_n = y_n$, by the A_γ-compactness of T at 0 some subsequence $V_{n_k}(x_{n_k}) \to x_0$ with $\mu x_0 \in T(x_0)$. Since $\|V_n(x_n)\| \leqslant C_1\|x_n\| \leqslant C_1 M$, $x_0 \in \bar{B}(0, C_1 M)$. Arguing by contradiction, the second assertion follows easily from the unique solvability. □

THEOREM 2. *Let X, K and K_n be as in Theorem 1 and $T\colon K \to 2^K$ A_γ-compact at 0 with $T_n\colon K_n \to CK(K_n)$ such that:*

(1) *there exists a $C_2 > 0$ such that $\|V_n(x)\| \geqslant C_2\|x\|$ for all n and all $x \in X_n$;*

(2) *there exists an $r_0 > 0$ such that T satisfies the Leray-Schauder condition (μ) on $\partial_K(B(0, r) \cap K)$ for all $r \in [r_0, C_1 r_0/C_2]$ and some positive $\mu \geqslant \gamma$;*

(λ) *there exists a $C_3 > 0$ such that if $\lambda x \in T_n(x)$ for some $x \in \partial_{K_n}(B_n(0, r_0/C_2) \cap K_n)$ and all n, then $\lambda \leqslant C_3$.*

Then the equation $\mu x \in T(x)$ is feebly approximation solvable in $B(0, C_1 r_0/C_2) \cap K$ with approximate solutions lying in $B_n(0, r_0/C_2) \cap K_n$ and is strongly approximation solvable if uniquely solvable.

PROOF. In view of Theorem 1, it suffices to show that T_n satisfies the Leray-Schauder condition on $\partial_{K_n}(B_n(0, r_0/C_2) \cap K_n)$ for all large n. We proceed by contradiction. Suppose that for some sequences $\{x_{n_k} | x_{n_k} \in \partial_{K_{n_k}}(B(0, r_0/C_2) \cap K_{n_k})\}$ and $\{\lambda_{n_k} | \lambda_{n_k} \geqslant \mu\}$ we have $\lambda_{n_k} x_{n_k} \in T_{n_k}(x_{n_k})$. By ($\lambda$) we may assume that $\lambda_{n_k} \to \lambda \in [\mu, C_3]$ which, together with $\|x_{n_k}\| = r_0/C_2$, implies that

$$\|V_{n_k}(\lambda_{n_k} x_{n_k} - \lambda x_{n_k})\| = |\lambda_{n_k} - \lambda| \cdot \|V_{n_k}(x_{n_k})\| \leqslant |\lambda_{n_k} - \lambda| \cdot C_1 r_0/C_2 \to 0$$

theorem $i_{K_n}((1/\mu)T_n, U_n) = i_{K_n}((1/\mu)T_n + \lambda h_n, U_n) \neq 0$. Thus, there exists $x_\lambda \in U_n$ such that $x_\lambda \in (1/\mu)T_n x_\lambda + \lambda h_n$ for each $\lambda > 0$. Take $\lambda_k \to \infty$ and observe that $\lambda_k h_n \in (x_{\lambda_k} - (1/\mu)T_n x_{\lambda_k})$ with $\lambda_k \|h_n\| \to \infty$ as $k \to \infty$. This contradicts the boundedness of $(I - (1/\mu)T_n)(U_n)$ and consequently, $i_{K_n}((1/\mu)T_n, U_n) = 0$. Next, using the homotopy $F_n(t, x) = (t/\mu)T_n(x)$, $x \in D_n$, we get that $i_{K_n}((1/\mu)T_n, D_n) = 1$ and by the additivity theorem $i_{K_n}((1/\mu)T_n, U_n \setminus \overline{D}_n) = -1$. Thus, for all large n there exists $x_n \in (U_n \setminus \overline{D}_n) \cap K_n$ such that $\mu x_n \in T_n(x_n)$. Since $\|x_n\| \leqslant M_1$ for all n, the A_γ-compactness of T at 0 imply that some subsequence $V_{n_k}(x_{n_k}) \to x_0$ with $\mu x_0 \in T(x_0)$. By condition (1), $x_0 \neq 0$. □

The following result provides some conditions on T and T_n which imply conditions (μ) and (β_n).

THEOREM 5. *Let X, K and K_n be as in Theorem* 1 *and T: $K \to 2^K$ be A_γ-compact with T_n: $K_n \to CK(K_n)$. Suppose that conditions* (1), (2) *and* (λ) *of Theorem* 2 *hold and that there exists an $R_0 > C_1 r_0/C_2$ such that for each $R \in [R_0, C_1 R_0/C_2]$ we have*

(β) there exists $0 \neq h_0 \in K$ such that $x \notin (1/\mu)T(x) + \beta h_0$ for all $x \in \partial_K(B(0, R) \cap K)$ and $\beta \geqslant 0$;

(τ) there exist a $C_4 > 0$ and $0 \neq h_n \in K_n$ with $V_n(h_n) \to h_0$ such that if $x \in (1/\mu)T_n(x) + \tau h_n$ for $x \in \partial_{K_n}(B_n(0, R_0/C_2) \cap K_n)$ and all n, then $\tau \leqslant C_4$.

Then the equation $\mu x \in T(x)$ is feebly approximation solvable in $B(0, C_1 R_0/C_2) \setminus B(0, r_0)$.

PROOF. As in Theorem 2 we get an $n_0 \geqslant 1$ such that T_n satisfies the Leray-Schauder condition (μ) on $\partial_{K_n}(B_n(0, r_0/C_2) \cap K_n)$ for each $n \geqslant n_0$. To show that (β_n) of Theorem 4 holds on $\partial_{K_n}(B_n(0, R_0/C_2) \cap K_n)$ for all $n \geqslant n_1$ $(\geqslant n_0)$, we argue by contradiction. If such an n_1 did not exist, we could find $\{x_{n_k} | x_{n_k} \in \partial_{K_{n_k}}(B_{n_k}(0, R_0/C_2) \cap K_{n_k})\}$ and $\{\beta_{n_k} \geqslant 0\}$ such that $x_{n_k} \in (1/\mu)T_{n_k}(x_{n_k}) + \beta_{n_k} h_{n_k}$. By (τ) we can assume that $\beta_{n_k} \to \beta \in [0, C_4]$ and let $y_{n_k} \in T_{n_k}(x_{n_k})$ be such that $x_{n_k} = (1/\mu)y_{n_k} + \beta_{n_k} h_{n_k}$. Then $V_{n_k}(y_{n_k} - \mu x_{n_k}) = -\mu\beta_{n_k} V_{n_k}(h_{n_k}) \to -\mu\beta h_0$, and by the A_γ-compactness of T, some subsequence $V_{n_{k(i)}}(x_{n_{k(i)}}) \to x_0$ with $-\mu\beta h_0 \in T(x_0) - \mu x_0$ and $R_0 \leqslant \|x_0\| \leqslant C_1 R_0/C_2$ by the properties of $\{V_n\}$. Thus, $x_0 \in (1/\mu)T(x_0) + \beta h_0$ with $\beta \geqslant 0$ and $R_0 \leqslant \|x_0\| \leqslant C_1 R_0/C_2$, in contradiction to (β) and consequently, an n_1 with the above property exists. The conclusion of our theorem now follows from Theorem 4. Moreover, since the approximate solutions $x_n \in B_n(0, R_0/C_2) \setminus \overline{B}_n(0, r_0/C_2)$, we have $r_0 < C_2\|x_n\| \leqslant \|V_n(x_n)\| \leqslant C_1\|x_n\| < C_1 R_0/C_2$ and consequently, the solution x_0 of $\mu x \in T(x)$ satisfies $r_0 \leqslant \|x_0\| \leqslant C_1 R_0/C_2$. □

As a consequence of Theorems 4 and 5, we have

COROLLARY 6 [7]. *Let X be a Banach space with a projectionally complete scheme Γ, $K \subset X$ a cone and $D_1 \subset D_2 \subset X$ two open and bounded sets with $0 \in D_1$. Suppose that T: $\overline{D}_2 \cap K \to 2^K$ is P_γ-compact with $P_n T$: $\overline{D}_2 \cap K_n \to$*

Corollary 1 *with* $\mu \geqslant 1$, *the equation* $Tx + Fx = \mu x$ *is f.a. solvable.*

(b) *If* $k_1 = 1 - k$ *with* $k \leqslant 1$ *in* (a) *and also* $(\mu I - T - F)(\bar{D} \cap K)$ *is closed, the equation* $Tx + Fx = \mu x$ *is solvable.*

(c) *In particular, if* $T = I - A$ *in* (a) (*or* (b)) *with* $A\colon X \to X$ *c-strongly accretive,* $k < c$ *or* F *is ball-condensing if* $c = 1$ (*or* $k = c$ *with* $c \geqslant 0$), $(I - A - F)(K) \subset K$ *and* $I - A - F$ *satisfies conditions* (1) *and* (2) *of Corollary* 1 *with* $\mu = 1$ (*and* $(A + F)(\bar{D} \cap K)$ *is closed*), *then* $Ax + Fx = 0$ *is f.a. solvable* (*solvable*).

In case when D does not contain 0, we have

THEOREM 3 [7]. *Let* X *be a Banach space with a projectionally complete scheme* Γ *and* $D \subset X$ *open and bounded. Suppose that* $T\colon \bar{D} \to 2^X$ *is* P_1-*compact with* $P_n T\colon \bar{D} \cap X_n \to CK(X_n)$ *u.s.c. and satisfies*:

(1) *if for some* $x_0 \in D$, $\lambda(x - P_n x_0) \in P_n T(x) - P_n(x_0)$ *for some* $x \in \partial D_n$ *and all large n, then* $\lambda \leqslant C$ *for some* $C > 0$;

(2) $\alpha(x - x_0) \notin T(x) - x_0$ *for* $x \in \partial D$ *and* $\alpha > 1$.

Then the equation $x \in T(x)$ *is feebly approximation solvable.*

The proof of this theorem is based on the degree theory for multivalued compact mappings. The single-valued case of Theorem 3 is a slightly corrected version of Theorem 4.4 in **[13]**. In particular, we have

COROLLARY 5. *Let* X, D *and* Γ *be as in Theorem* 3, $\|P_n\| \leqslant 1$, $T\colon X \to X$ *and* $F\colon \bar{D} \to X$ *as in Corollary* 4 (a) (*or* (c)) *and* $T + F(I - A - F)$ *satisfies conditions* (1) *and* (2) *of Theorem* 3 *with* $\mu = 1$.

Then the equation $Tx + Fx = x$ ($Tx + Fx = 0$) *is f.a. solvable.*

This result could be also extended to cover the cases (b) and (c) with $k = c \geqslant 0$ in Corollary 4.

Finally, we now provide some results involving nonzero solutions of $\mu x \in T(x)$.

THEOREM 4. *Let* X, K, K_n, D *and* D_n *be as in Theorem* 1 *and condition* (1) *of Theorem* 2 *hold. Let* $U \subset X$ *and* $U_n \subset X_n$ *be open and such that* $D \subset U$, $B_n(0, r) \subset D_n \subset U_n$ *for some* $r > 0$ *and* $\{U_n\}$ *uniformly bounded by some constant* $M_1 > 0$. *Suppose that* $T\colon \bar{U} \cap K \to 2^K$ *is* A_γ-*compact at* 0 *and satisfies condition* (μ) *of Theorem* 1 *on* $\partial_{K_n}(D_n \cap K_n)$ *and for each large n*

(β_n) *there exists* $0 \neq h_n \in K_n$ *such that*

$$x \notin (1/\mu)T_n(x) + \beta h_n \quad \text{for all } \beta \geqslant 0 \text{ and } x \in \partial_{K_n}(U_n \cap K_n).$$

Then the equation $\mu x \in T(x)$ *is feebly approximation solvable in* $U \cap K \setminus \{0\}$.

PROOF. We first show that for all large n the fixed point index $i_{K_n}((1/\mu)T_n, U_n) = 0$. Suppose that for some large n, $i_{K_n}((1/\mu)T_n, U_n) \neq 0$ and define $H_n\colon [0, 1] \times (\bar{U}_n \cap K_n) \to CK(K_n)$ by $H_n(t, x) = (1/\mu)T_n x + t\lambda h_n$ with $\lambda > 0$. Then $x \notin H_n(t, x)$ for $t \in [0, 1]$ and $x \in \partial_{K_n}(U_n \cap K_n)$ and by the homotopy as $k \to \infty$. By the A_γ-compactness of T at 0, a subsequence $V_{n_{k(i)}}(x_{n_{k(i)}}) \to x_0$ with $\lambda x_0 \in T(x_0)$, and the properties of V_n imply that $r_0 \leqslant \|x_0\| \leqslant C_1 r_0/C_2$, in contradiction to (2). The conclusions of our theorem now follow from Theorem 1 by taking $B(0, r_0/C_2)$ and $B_n(0, r_0/C_2)$ as D and D_n. □

REMARK. Condition (λ) in Theorem 2 is satisfied if for each real number R there is a constant $C(R) > 0$ such that $\|u\| \leqslant C(R)$ for all $u \in T_n(x)$ with $\|x\| = R$, $x \in K_n$ and all n.

REMARK. When $K = X$, $K_n = X_n$, $\mu = 1$, D and D_n are balls and T and T_n single-valued, Theorems 1 and 2 were proved by Lees-Schultz **[6]** for their A_1-compact mappings and extend a theorem of Petryshyn **[10]**. We add that their arguments required the convexity of D and D_n.

COROLLARY 1 [7]. *Let X be a Banach space with a projectionally complete scheme $\Gamma = \{X_n, V_n, P_n\}$, $K \subset X$ a cone, $K_n = K \cap X_n$ and $D \subset X$ open and bounded with $0 \in D$. Let T: $\bar{D} \cap K \to 2^K$ be P_γ-compact at 0 with $P_n T$: $\bar{D} \cap K_n \to CK(X_n)$ u.s.c. and $P_n(K) \subset K$ for all large n. Suppose that*:

(1) *T satisfies the Leray-Schauder condition (μ) on $\partial_K(D \cap K)$;*

(2) *$P_n T$ satisfies condition (λ) of Theorem 2 on $\partial_{K_n}(D_n \cap K_n)$ for all large n.*

Then the equation $\mu x \in T(x)$ is feebly approximation solvable in $D \cap K$ and strongly approximation solvable if uniquely solvable.

As consequences of this result we just mention here the following ones (many others can be found in [7]).

COROLLARY 2 [7]. *Let X be a reflexive π_1-Banach space, K, D and Γ as in Corollary 1. Suppose that T: $K \to CK(K)$ is a generalized contraction (i.e., for each $x \in K$ there exists $\alpha(x) \in (0, 1)$ such that $\delta(Tx, Ty) \leqslant \alpha(x)\|x - y\|$ for all $x, y \in K$, where δ is the Hausdorff distance induced by the norm of X) and C: $\bar{D} \to CK(K)$ demiclosed and compact. Then, if $T + C$ satisfies condition (μ) on $\partial_K(D \cap K)$ for some $\mu \geqslant 1$, the equation $\mu x \in T(x) + C(x)$ is feebly approximation solvable and is strongly approximation solvable if uniquely solvable.*

COROLLARY 3 [7]. *Let X be a reflexive π_1-Banach space and T: $X \to CK(X)$ a generalized contraction. Then for each $f \in X$ the equation $f \in x - Tx$ is feebly approximation solvable in $B(0, r_f)$ with $r_f = \max\{\|v + f\|: v \in T(0)\}/(1 - \alpha(0))$.*

For the single-valued case with $K = X$ and T: $X \to X$ or T: $\bar{D} \to \bar{D}$ see **[1]** and the references therein. Moreover if T is k-strict contractive, $k < 1$, Corollaries 2 and 3 hold without the reflexivity of X (cf. also **[14]**).

COROLLARY 4. (a) *Let X, K, D and Γ be as in Corollary 1, $\|P_n\| \leqslant 1$, T: $X \to X$ continuous k-strongly pseudo-contractive, i.e. $(Tx - Ty, x - y)_- = \inf\{(Tx - Ty, u) | u \in J(x - y)\} \leqslant k\|x - y\|^2$ for x, y in X, with $T(K) \subset K$ and F: $\bar{D} \cap K \to K$ continuous and k_1-ball-contractive with $k_1 < 1 - k$ or ball-condensing if $k = 0$. Then, if $T + F$ satisfies conditions* (1) *and* (2) *of*

$CK(X_n)$ u.s.c. and $P_n(K) \subset K$ for all large n. Suppose also that T satisfies conditions (μ) and (λ) on $\partial_K(D_1 \cap K)$ and $\partial_{K_n}(D_1 \cap K_n)$ respectively and conditions (β) and (τ) on $\partial_K(D_2 \cap K)$ and $\partial_{K_n}(D_2 \cap K_n)$ respectively. Then the equation $\mu x \in T(x)$ is feebly approximation solvable in $(D_2 \setminus \overline{D}_1) \cap K$.

REMARK. Theorem 4 is valid if condition (2) holds on $\partial_{K_n}(U_n \cap K_n)$ and condition (β_n) holds on $\partial_{K_n}(D_n \cap K_n)$, while Theorem 5 is also valid if condition (β) holds on $\partial_K(B(0, r) \cap K)$ for each $r \in [r_0, C_1 r_0/C_2]$, (τ) holds on $\partial_{K_n}(B_n(0, r_0/C_2) \cap K_n)$, (μ) holds on $\partial_K(B(0, R) \cap K)$ for each $R \in [R_0, C_1 R_0/C_2]$ and (λ) holds on $\partial_{K_n}(B_n(0, R_0/C_2) \cap K_n)$. The same observation applies to Corollary 4. For some existence results for positive single-valued P-compact mappings see **[4]**, **[5]** (see also **[17]**, **[2]** and the references therein). In particular, Corollary 6 is applicable to the mappings considered in Corollary 4.

Analyzing the proofs of Theorems 1 and 4, we see that just the solvability assertion for $\mu x \in T(x)$ still holds for the following larger class of mappings.

DEFINITION 4. A multivalued mapping T: $D \subset X \to 2^X$ is said to be *pseudo-A_γ-compact* if condition (1) of Definition 2 holds and there exists $\gamma \geqslant 0$ such that if for some positive $\mu \geqslant \gamma$ and a sequence $\{x_{n_k} | x_{n_k} \in D_{n_k}\}$ with $\|x_{n_k}\| \leqslant M$ for all n and some $M > 0$ we have that $V_{n_k}(y_{n_k} - \mu x_{n_k}) = 0$ for some $y_{n_k} \in T_{n_k}(x_{n_k})$, then there exists $x \in D$ such that $\mu x \in T(x)$.

If $\Gamma = \{X_n, V_n, P_n\}$ is a projectionally complete scheme for X, then we can take $P_n T$ as T_n, in which case we obtain the so-called G-mappings of Figueiredo **[3]** with T single-valued and $\mu = 1$. Various examples of such mappings (some of which are not P_1-compact) can be found in **[3]**. We add that for quite many of these examples one can show that the above sequence $\{x_{n_k}\}$ in Definition 4 has a subsequence weakly converging to a solution of $\mu x \in T(x)$ (see **[12]**, **[8]** where a related notion of pseudo A-properness is considered). Finally, using the degree theory, one can establish the following:

THEOREM 6. *Let X and Γ be as in Corollary 4, T: $X \to X$ continuous and F: $\overline{B}(0, r) \subset X \to X$ demicontinuous and k-ball-contractive such that either $(I - T - F)(\partial B) \subset B$ or $(Tx + Fx, u) \geqslant 0$ for $u \in J(x)$, $x \in \partial B$.*

(a) *If T is surjective, a-stable and either $k < c$ or F is ball-condensing if $c = 1$, the equation $Tx + Fx = 0$ is f.a. solvable.*

(b) *If $T_\varepsilon = T + \varepsilon I$ is surjective for $\varepsilon > 0$, T and T_ε are a-stable with $c_\varepsilon > c$ (c_ε is the constant of a-stability of T_ε), $k = c$ and $(T + F)(\overline{B})$ is closed, then $Tx + Fx = 0$ is solvable.*

(c) *If T_ε is a-stable and surjective for $\varepsilon > 0$ with $c_\varepsilon \to 0$ as $\varepsilon \to 0$, F is compact and $(T + F)(\overline{B})$ is closed, then $Tx + Fx = 0$ is solvable.*

REMARK. As T and F in Theorem 6 one can take the mappings A and F considered in Corollary 4 (c) or $T = I - A$ with A and F as in Corollary 4(a) and (b) with either $(Ax + Fx, u) \leqslant \|x\|^2$ for $u \in J(x)$, $x \in \partial B$ or $(I - A - F)(\partial B) \subset B$. Here and in all above results one can allow F to be multivalued.

This theory of k-ball-contractive perturbations of (strongly) pseudo-contractive and accretive mappings extends the corresponding theory of Browder **[14]** involving compact perturbations. Detailed proofs of these and other results will be given elsewhere. Various surjectivity results for ball-condensing perturbations of these mappings can be found in **[8]**, **[15]**, **[16]**.

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