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A SIMPLE PROOF OF A COVERING PROPERTY OF LOCALLY COMPACT GROUPS

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ABSTRACT. We give a simple proof of the following result of Emerson and Greenleaf.

THEOREM. *Let V be a relatively compact subset with nonvoid interior of a locally compact group G. Then there exist a subset $T \subset G$ and a natural number M such that $G = \bigcup_{t \in T} tV$ and at most M of the tV's, $t \in T$, intersect.*

The result cited above is proved in **[4]** and is used there and in **[2]** in the course of proving that every amenable locally compact group G has strong properties such as

(A) If $\varepsilon > 0$ and compact $K \subset G$ containing the identity of G are given, there is a compact $U \subset G$ with $|U| > 0$ such that $|KU \,\Delta\, U|/|U| < \varepsilon$.

(Here $|U|$ indicates left Haar measure of the set U. And we remind the reader that G is called amenable if $L^\infty(G)$ admits a left invariant mean; see **[6]**, **[8]**, **[1]** for further details.)

The proof given in **[4]** of the theorem above involves some delicate arguments about geometry of groups. We discovered the simple proof presented below in the course of preparing **[1]** (and were apprised later that it is almost the same as a proof in Chapter 8, §1.7, of **[7]**); our reason for publishing it now is that it seems not widely known, according to **[3]**, **[5]**, that such a proof exists.

PROOF OF THE THEOREM. After a reduction as in [**2**; Proposition 2], we are left with the task of taking a relatively compact symmetric neighbourhood V of $e \in G$ and finding $T \subset G$ and constant M so that $G = \bigcup_{t \in T} tV$ and at most M of the tV's, $t \in V$, intersect. We may assume the open and closed subgroup $\bigcup_1^\infty V^n$ of G equals G. (For, if we cover $\bigcup_1^\infty V^n$ with $\bigcup_{t \in T} tV$, then $\bigcup_{t \in T} stV$ covers the coset $s \bigcup_1^\infty V^n$ and hence we cover the whole group.) And we may assume the subgroup $\bigcup_1^\infty V^n$ is not compact. (Otherwise we can cover it with a finite number of left translates of V and proceed as in the previous parenthetical remark.) We then get our set $T \subset G$ as follows.

Let $t_1 = e$. Since G is not compact, $V^2 \neq V$ and there is a $t_2 \in (V^2)^- \setminus V$.

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If $(V^2)^- \setminus \bigcup_1^2 t_i V \neq \varnothing$, take t_3 in it. Continuing like this, we get $(V^2)^- \subset \bigcup_1^{N_2} t_i V$ with

$$t_j \notin \bigcup_1^{j-1} t_i V, \qquad 2 \leqslant j \leqslant N_2.$$

(Note that, if W is a symmetric neighbourhood of e such that $W^2 \subset V$, then $N_2 \leqslant |(V^2)^- W|/|W|$.) If $(V^3)^- \setminus \bigcup_1^{N_2} t_i V \neq \varnothing$, choose t_{N_2+1} in it. And so on. Hence, by induction, we get $(V^n)^- \subset \bigcup_1^{N_n} t_i V$ with

$$t_j \notin \bigcup_1^{j-1} t_i V, \qquad 2 \leqslant j \leqslant N_n;$$

thus $G = \bigcup_1^\infty t_i V$ (and $N_n \leqslant |(V^n)^- W|/|W|$). Suppose $s \in t_i V$. Then $t_i \in sV$ and $t_i W \in sVW$ (where $W^2 \subset V$ as above). And, if s is also in $t_j V$, then $t_j W \subset sVW$ with $t_i W \cap t_j W = \varnothing$ if $i \neq j$. It follows that s is contained in at most $|VW|/|W|$ of the $t_i V$'s, $i = 1, 2, 3, \ldots$.

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