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ON K-SEMIMETRIC SPACES

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ABSTRACT. An example is constructed of a separable Moore space that does not possess a compatible K-semimetric.

A semimetric d for a space X is said to be a K-semimetric if $d(H, K) > 0$ whenever H and K are disjoint compact subsets of X. It is the purpose of this note to provide an example of a regular semimetrizable space (in fact, a separable Moore space) which does not have a compatible K-semimetric. This answers a question first posed by A. V. Arhangel'skiĭ in [**1**] and later by others such as H. Martin in [**3**].

The description of the example follows below. The sets R, P, Q, and N denote the real numbers, irrational numbers, rational numbers, and natural numbers respectively.

EXAMPLE 1. A separable Moore space which is not K-semimetrizable.

Let $A_1 = P \times \{0\}$, $A_2 = P \times \{-1\}$, $E = \{(r, s) \in Q \times Q\colon s > 0\}$, and $X = A_1 \cup A_2 \cup E$. Describe a local base for points in X as follows: Points in E have the usual neighborhoods (as inherited from $R \times R$). If $a \in P$ and $n \in N$ let

$$U_n(a, 0) = \{(a, 0)\} \cup \{(r, s) \in E\colon a < r < s/n + a, s < 1/n\},$$

$$U_n(a, -1) = \{(a, -1)\} \cup \{(r, s) \in E\colon -s/n + a < r < a, s < 1/n\}.$$

Then $\{U_n(a, 0)\}_1^\infty$, and $\{U_n(a, -1)\}_1^\infty$, give local bases at $(a, 0) \in A_1$ and $(a, -1) \in A_2$ respectively. (A simple sketch reveals that $U_n(a, 0)$ is $(a, 0)$ along with the "right half of the interior of a V neighborhood at $(a, 0)$" and $U_n(a, -1)$ is $(a, -1)$ along with the "left half".) It is easily verified that X (with the new topology) is a separable, completely regular Moore space.

Let d be a semimetric for X–we show that d is not a K-semimetric.

For $n \in N$ let

$$P(n) = \{a \in P\colon d(x, y) \geqslant 1/n, \text{all } x \in U_n(a, 0), y \in U_n(a, -1)\}.$$

If $P = \bigcup_{n=1}^\infty P(n)$ there is some $k \in N$ such that $T = \operatorname{int}_R(\operatorname{cl}_R, P(k)) \neq \emptyset$. It is possible to find $(t_1, s), (t_2, s) \in E \cap (T \times R)$ and $b \in P(k)$ such that

$$d((t_1, s), (t_2, s)) < \frac{1}{k}, \quad |t_1 - t_2| < \frac{1}{k(k+1)}, \quad s = \frac{1}{k+1}$$

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and $t_1 < b < t_2$. If $x = (t_2, s), y = (t_1, s)$ then $x \in U_k(b, 0), y \in U_k(b, -1)$ and $d(x, y) < 1/k$ which contradicts the definition of $P(k)$. It follows that $\bigcup_{n=1}^{\infty} P(n) \neq P$ and there exists some

$$a \in P - \left(\bigcup_{n=1}^{\infty} P(n) \right).$$

Now $a \notin P(n)$ implies there exists $x_n \in U_n(a, 0)$ and $y_n \in U_n(a, -1)$ such that $d(x_n, y_n) < 1/n$. Clearly $x_n \to (a, 0)$ and $y_n \to (a, -1)$. If $H = \{x_n\}_1^{\infty} \cup \{(a, 0)\}$ and $K = \{y_n\}_1^{\infty} \cup \{(a, -1)\}$ then K and H are disjoint compact subsets of X, but $d(H, K) = 0$; thus d is not a K-semimetric for X and the proof is complete.

REMARKS 2. (a) It is trivial that every metric space is K-semimetrizable. In fact, A. V. Arhangel'skiĭ [1] has shown that a regular space Y is metrizable if and only if there is a compatible semimetric d for Y such that $d(A, B) > 0$ whenever A and B are disjoint subsets of Y with A compact and B closed.

(b) It is known [1] that a semimetric space which is submetrizable (has a weaker metric topology) is also K-semimetrizable. W. Lindgren has pointed out to the author that a semimetric space with a coarser T_2 quasimetric topology is K-semimetrizable and that apparently, Example 1 gives the first known example of a Moore space that does not admit a coarser T_2 quasimetric topology.

(c) Let X be the space of Example 1 and let Y be the quotient space obtained from X by identifying points $(a, 0)$ and $(a, -1)$ for each $a \in P$. Then Y is a completely regular separable submetrizable Moore space. If f: $X \to Y$ is the corresponding quotient mapping then f is a perfect map from the nonsubmetrizable space X onto a submetrizable Moore space Y. This should be contrasted with the result by Borges [2] and Okuyama [4] that if g: $Z \to M$ is a perfect map from a Hausdorff space Z onto a metric space M then Z is metrizable if and only if Z has a G_δ-diagonal. This suggests the following question.

QUESTION 1. If g: $Z \to M$ is a perfect map from a regular space Z onto a submetrizable space M what minimal diagonal condition on Z will ensure that Z is submetrizable? G. M. Reed has an example [6, Example 3] that shows Z need not be submetrizable even if Z has a regular G_δ-diagonal.

Besides the submetrizable condition mentioned in Remark 2(b) different authors have given various sufficient conditions for a Moore space to be K-semimetrizable. H. Martin showed that a locally connected rim compact space is K-semimetrizable if and only if it is a developable γ-space [3]. A result by P. Zenor shows that a Moore space with a regular G_δ-diagonal is K-semimetrizable [5].

QUESTION 2. What minimal topological condition on a Moore space (or semimetric space) will ensure that the space be K-semimetrizable? For example, is every locally connected rim compact Moore space necessarily K-semimetrizable?

References

1. A. V. Arhangel'skiĭ, *Mappings and spaces*, Uspehi Mat. Nauk **21** (1966), no. 4 (130), 133–184 = Russian Math. Surveys **21** (1966), no. 4, 115–162. MR **37** #3534.

2. C. R. Borges, *On stratifiable spaces*, Pacific J. Math. **17** (1966), 1–16.

3. H. W. Martin, *Local connectedness in developable spaces*, Pacific J. Math. **61** (1975), 219–224.

4. A. Okuyama, *On metrizability of M-spaces*, Proc. Japan Acad. **40** (1964), 176–179.

5. P. L. Zenor, *On spaces with regular G_δ-diagonals*, Pacific J. Math. **40** (1972), 759–763.

6. G. M. Reed, *Some Moore examples*, Notices Amer. Math. Soc. **22** (1975), Abstract 726-54-3, p. A-582.

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