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COMPACT-OPEN VERSUS k-COMPACT-OPEN

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ABSTRACT. We obtain various examples of k-spaces, one of which is first countable and another is compact, such that the space of continuous functions from a compact metric space to any of these spaces, with the compact-open topology, is not a k-space. We also improve some results on products of k-spaces.

1. Introduction. For any spaces X and Y, let Y^X denote the set of continuous functions from X to Y. Let co denote the compact-open topology and $k(\mathrm{co})$ denote the finest topology on Y^X which agrees with co on every compact subset of (Y^X, co). If Y is dominated by a family $\mathfrak{F} = \{Y_\alpha\}_{\alpha\in\Lambda}$ of subspaces (i.e., $A \subset Y$ is closed if and only if, for some subfamily $\mathcal{G}$ of $\mathfrak{F}$ which covers A, $A \cap X_\beta$ is closed in X_β, for each $X_\beta \in \mathcal{G}$; without loss of generality, we assume that $\mathfrak{F}$ is closed with respect to finite unions) then let $w(\mathrm{co})$ denote the weak topology on Y^X with respect to the family $\{(Y_\alpha^X, \mathrm{co})\}_{\alpha\in\Lambda}$. For convenience, let $(Y^X, \mathrm{co}) \equiv Y^X$, $(Y^X, k(\mathrm{co})) \equiv k(Y^X)$ and $(Y^X, w(\mathrm{co})) = w(Y^X)$.

It is well known that if Y is metrizable and X is compact Hausdorff then Y^X is metrizable and therefore $Y^X = k(Y^X)$. In §2 we show that, even for spaces with very strong separation properties, short of metrizability, the preceding equality fails.

We conclude this section with three useful results which appear to be folklore, but none seems to be recorded elsewhere. Throughout, all spaces are *assumed* to be *Hausdorff*.

LEMMA 1.1. *Let X be dominated by a family $\mathfrak{F} = \{X_\alpha\}_{\alpha\in\Lambda}$ of subspaces. Pick sequence $\{X_1, X_2, \dots\} \subset \mathfrak{F}$ and sequence $\{w_1, w_2, \dots\}$ such that $w_n \in X_n - X_{n-1}$. Then the set $A = \{w_n | n = 1, 2, \dots\}$ is a closed and discrete subspace of X.*

PROOF. Observe that each $A \cap X_n$ is a finite, and thus closed, subset of X_n and $A \subset \bigcup_{n=1}^\infty X_n$. Therefore A is closed, since $\mathfrak{F}$ dominates X. Similarly one can prove that each $A_i = A - \{w_i\}$ is a closed subset of X, and thus of A. Therefore A is discrete, which completes the proof.

LEMMA 1.2. *Let C be a compact space and let X be domainted by a family*

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$\mathcal{F} = \{X_\alpha\}_{\alpha\in\Lambda}$ of subspaces. If K is a compact subset of X^C then $K \subset K^C_\alpha$, for some $\alpha \in F$.

PROOF. Suppose not (note that $\mathcal{F}$ is assumed to be closed with respect to finite unions). Then there exists a countable subfamily $\{X_{\alpha_n} | \alpha_n \in \Lambda,\ n = 1, 2, \ldots\}$ such that

(a) $X_{\alpha_1} \subset X_{\alpha_2} \subset \ldots$,

(b) there exists $f_n \in (X^C_{\alpha_n} - X^C_{\alpha_{n-1}}) \cap K$.

Therefore, for each n, there exists $x_n \in C$ such that

$$f_n(x_n) \in X_{\alpha_n} - X_{\alpha_{n-1}}.$$

Since $K \times C$ is compact, we get that $\{(f_n, x_n)\}$ has some cluster point (f, x). Since the evaluation map ω: $X^C \times C \to X$ is continuous, then $f(x)$ is a cluster point of $\{f_n(x_n)\}$. Consequently $f(x)$ is a limit point of $A = \{f_1(x_1), f_2(x_2), \ldots\}$, since the elements of A are all distinct. However, A is a closed and discrete subset of X, by Lemma 1.1. Therefore A can have no limit point. We have thus obtained a contradiction, which completes the proof.

PROPOSITION 1.3. *Let X be dominated by a family $\{X_\alpha\}_{\alpha\in\Lambda}$ of subspaces. If C is compact, then w(co) $\subset k$(co). On the other hand, if for each $\alpha \in \Lambda$, X^C_α is a k-space, then k(co) $\subset w$(co).*

PROOF. It is easily seen that w(co) $\subset k$(co). For any closed subset A of $w(X^C)$, each $A \cap X^C_\alpha$ is closed in X^C_α. Therefore, by Lemma 1.2, one immediately gets that $A \cap K$ is closed in K for each compact subset K of X^C, which implies that A is a closed subset of $k(X^C)$.

Finally we show that k(co) $\subset w$(co). Let A be a closed subset of $k(X^C)$. Then, for each α, $A \cap K$ is closed in K for each compact subset K of X^C_α. Therefore, since X^C_α is a k-space, $A \cap X^C_\alpha$ is closed in X^C_α for each α, and hence A is a closed subset of $w(X^C)$. This completes the proof.

2. Examples. For any collection $\{I_\alpha\}_{\alpha\in\Lambda}$ of closed unit intervals (throughout, Λ will be assumed to be an ordinal number) let K_Λ denote the quotient space of the disjoint topological union $\bigvee_{\alpha\in\Lambda} I_\alpha$ of the I_α which results from identifying the zeros of all the I_α. (K_Λ is the hedgehog simplicial complex with the CW-topology.)

EXAMPLE 2.1. If card $\Lambda = 2^{\aleph_0}$ then $K_\Lambda \times K_\Lambda = K^2_\Lambda$ is not a k-space. Furthermore, assuming the continuum hypothesis, K^2_Λ does not have the weak topology over $\{K^2_\alpha\}_{\alpha\in\Lambda}$.

PROOF. In the Example on p. 563 of [**4**], it is proved that $K_\Lambda \times K_{\omega_0}$ is not a k-space. Since $K_\Lambda \times K_{\omega_0}$ is a closed subspace of $K_\Lambda \times K_\Lambda$, it follows that $K_\Lambda \times K_\Lambda$ is not a k-space.

Assuming the continuum hypothesis, we get that card $\alpha = \aleph_0$ for each $\alpha \in \Lambda$; consequently each K^2_α is a k-space, by Lemma 8.1 of [**5**]. Since the weak topological union of k-spaces is a k-space, it follows that K^2_Λ does not have the weak topology over $\{K^2_\alpha\}_{\alpha\in\Lambda}$.

EXAMPLE 2.2. There exists a stratifiable $\aleph_0$-space Y such that $Y = \Sigma_n Y_n$, with each Y_n separable metrizable, but $Y^2 \neq \Sigma_n Y_n^2$ and Y^2 is not a k-space.

PROOF. Let Q be the space of rational numbers, Z the set of integers, f: $Q \to Q/Z$ the natural quotient map and 1: $Q \to Q$ the identity map. Dieudonné (see Example 1 on p. 130 of [3]) has shown that $f \times 1$: $Q \times Q \to Q/Z \times Q$ is not a quotient map even though f and 1 are closed continuous functions. Therefore $Q/Z \times Q$ is not a k-space (it is easily seen that $f \times 1$ is compact-covering; by Lemma 11.2 of [7], $Q/Z \times Q$ cannot be a k-space).

Now let $Y = Q/Z$. Clearly Y is a stratifiable $\aleph_0$-space and $Y = \Sigma_n Y_n$ such that $Y_n = f([-n, n] \cap Q)$ (indeed Y is homeomorphic to the subspace of K_{ω_0} which consists of the rational points of each spine). However, Y^2 is not a k-space, since $Q/Z \times]\frac{1}{4}, \frac{3}{4}[$ is a closed subspace of Y which is homeomorphic to $Q/Z \times Q$. From Proposition 1.3 we get that $Y^2 \neq \Sigma_n Y_n^2$. This completes the proof.

EXAMPLE 2.3. There exists a first countable, separable, σ-compact stratifiable space X such that X^I is not a k-space.

PROOF. Let $X = \{(x, y) | x$ and y are real numbers and $y \geqslant 0\}$ with the well-known "butterfly" topology (see p. 1075 of [1]). Let $A \subset X^I$ consist of all continuous functions f: $I \to X$ such that each $f(t) = (t, y_t)$ for some $y_t \geqslant 0$ and $\{t \in I | y_t = 0\}$ is a nowhere dense subset of I.

Now let K be a compact subset of X^I and let us show that $K \cap A$ is a closed subset of K. Let $\{f_\nu\}_{\nu \in \Gamma}$ be a net in $K \cap A$ which converges to some $f \in K$. Since f_ν converges to f pointwise it is immediate that each $f(t) = (t, y(t))$ for some $y(t) \geqslant 0$. (We will next show that $\{t \in I | y(t) = 0\}$ is a nowhere dense subset of I.) Suppose $\{t \in I | y(t) = 0\}$ is not a nowhere dense subset of I. Then there exists some $[c, d] \subset I$ such that $y(t) = 0$ for each $c \leqslant t \leqslant d$ (recall that $f(I)$ is compact). Then for each $c \leqslant t \leqslant d$ there exists some function f_ν such that $f_\nu(t) = (t, 0) = f(t)$, because of the butterfly neighborhoods of $(t, 0)$.

For $n = 1, 2, \ldots,$ let B_n be the set of all $c \leqslant t \leqslant d$ such that for some f_ν, $\nu \in \Gamma$, $f_\nu(t) = (t, y_\nu(t))$ with $0 < y_\nu(t) < 1/n$. Each B_n is open in $[c, d]$ (because the evaluation map ω: $X^I \times I \to X$ is continuous) and dense in $[c, d]$ (because, for each $c \leqslant t \leqslant d$, there exists some f_ν, $\nu \in \Gamma$, such that $f_\nu(t) = (t, 0)$). By the Baire Category Theorem, there exists $c < u < d$ such that some $f_{\nu_n}(u) = (u, y_{\nu_n}(u))$ with $0 < y_{\nu_n}(u) < 1/n$, for $n = 1, 2, \ldots.$ Therefore, by the continuity of the evaluation map, $\omega(K \times I)$ is compact but contains the closed and noncompact subset $\{(u, y_{\nu_n}(u)) | n = 1, 2, \ldots\}$, a contradiction. Consequently the function $f \in A \cap K$ and thus $A \cap K$ is closed; that is, $A \cap C$ is closed for any compact subset C of X^I. However, A is not a closed subset of X^I, since the function h: $I \to X$, such that $h(t) = (t, 0)$ for all $t \in I$, is not in A but is clearly in A^-. This completes the proof.

EXAMPLE 2.4. There exists a compact space Y such that Y^I is not a k-space.

PROOF. Let X be the butterfly space of Example 2.3 and $Y = BX$ be any

Hausdorff compactification of X. Using the same argument of Example 2.3 we immediately get that Y^I is not a k-space.

3. Positive results. The first two results are improvements of Lemma 8.1 on p. 194 of **[5]**. The fourth shows that indeed there are nonmetrizable spaces Y such that Y^C is first-countable (therefore a k-space), for any compact space C.

PROPOSITION 3.1. *Let* $X = \Sigma_n X_n$ *such that each* X_n *is locally compact and* $X_n \subset X_{n+1}$. *If* C *is a finite discrete space then* $k(X^C) = X^C = \Sigma_n X_n^C$.

PROOF. Clearly it suffices to prove this result for the case where C consists of two points. In this case, the proof of the second equality is the same as the proof of Lemma 8.1 of **[5]**, with some obvious changes. The first equality follows from Proposition 1.3.

PROPOSITION 3.2. *Let* X *be dominated by the point-countable family* $\{S_\alpha\}_{\alpha\in\Lambda}$ *such that each* X_α *is locally compact. If* C *is a finite discrete space then* $k(X^C) = X^C$.

PROOF. Again let C consist of two points. By the theorem in **[8]** we only need show that $X \times X$ is locally a k-space (i.e., each point in $X \times X$ has a neighborhood whose closure is a k-space). Note that if $x \in X$ and $x \in$ only $X_{\alpha_1}, X_{\alpha_2}, \ldots,$ then $x \in X - \bigcup\{X_\alpha | \alpha \neq \alpha_i$ for $i = 1, 2, \ldots\} \subset \bigcup_{i=1}^{\infty} X_{\alpha_i}$, which shows that $x \in (\bigcup_{i=1}^{\infty} X_{\alpha_i})^0$. Therefore, if $x \in$ only $X_{\alpha_1}, X_{\alpha_2}, \ldots, y \in$ only $X_{\beta_1}, X_{\beta_2}, \ldots,$ and $Z = \bigcup_{i=1}^{\infty} X_{\alpha_i} \cup \bigcup_{i=1}^{\infty} X_{\beta_i}$ then $(x, y) \in Z^0 \times Z^0 \subset Z \times Z$ which is closed in $X \times X$ and a k-space, by Proposition 3.1.

PROPOSITION 3.3. *Let* C *be compact and* Y *be covered by a closed locally finite collection* $\{Y_\alpha\}_{\alpha\in\Lambda}$ *such that, for each finite subcollection* F *of* $\{Y_\alpha\}_{\alpha\in\Lambda}$, $(\bigcup F)^C$ *is a* k-*space. Then* Y^C *is a* k-*space.*

PROOF. As indicated in the proof of Proposition 3.2, we only need show that Y^C is locally a k-space.

For each $y \in Y$, let $F_y = \{Y_\alpha | y \in Y_\alpha\}$. Clearly F_y is finite and $y \in (\bigcup F_y)^0$. Therefore, for $f \in Y^X$, the compact set $f(C)$ is contained in the interior of the union of some finite subcollection F_f of $\{Y\}_{\alpha\in\Lambda}$ and $(\bigcup F_f)^C$ is a k-space.

Proposition 3.3 becomes false if $\{Y_\alpha\}_{\alpha\in\Lambda}$ is assumed to be only hereditarily closure-preserving–consider the collection of finite subcomplexes of K_Λ in Example 2.1.

For our last result we need the concept of hemicompactness of a space C, which means that there exists a countable family of compact subsets of C such that every compact subset of C is contained in some member of this family.

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[1]We thank the referee for sharp improvements of our original Proposition 3.4 and for **[6]**.

PROPOSITION 3.4. *If C is hemicompact and Y has a point-countable base, then Y^C is first countable.*[1]

PROOF. Let $C = \bigcup_{n=1}^{\infty} C_n$, where each C_n is compact and each compact subset of C is contained in some C_n. Let $\mathfrak{B}$ be a point-countable base for Y. Let $f \in Y^C$. For each n, let $\mathfrak{B}_n' = \{B \in \mathfrak{B} \mid B \cap f(C_n) \neq \varnothing\}$ and $\mathcal{P}_n = \{(B, D) \in \mathfrak{B}_n' \times \mathfrak{B}_n' \mid \overline{D \cap f(C_n)} \subset B\}$. (From Prop. 2.1 of [2], note that each $\mathfrak{B}_n'$ and, therefore, each $\mathcal{P}_n$ is countable.) Then the basic open sets of the form $\bigcap_{i=1}^{k} \langle C_n \cap f^{-1}(\overline{D_i \cap f(C_n)}), B_i \rangle$, for all n and all finite subsets $\{(B_1, D_1), \ldots, (B_k, D_k)\}$ of $\mathcal{P}_n$, will be a countable neighborhood base for f. (Simply note that if $f \in \langle K, V \rangle$ then $K \subset$ some C_j; therefore, by the regularity of $f(C_j)$, there exists a finite subset $\{(B_1, D_1), \ldots, (B_k, D_k)\}$ of $\mathcal{P}_j$ such that $f(K) \subset \bigcup_{i=1}^{k} D_i$ and $\bigcup_{i=1}^{k} B_i \subset V$, which implies that $f \in \bigcap_{i=1}^{k} \langle C_j \cap f^{-1}(\overline{D_i \cap f(C_j)}), B_i \rangle \subset \langle K, V \rangle$.)

Essentially, the converse of Proposition 3.4 is also true. Indeed 1.5(c) of **[6]** states that *if C is completely regular and Y contains a nontrivial path, hemicompactness of C is a necessary condition for Y^C to be first countable.*

Example 6.6 of **[2]** is an example of a nonnormal space Y which satisfies the hypothesis of Proposition 3.4.

The example in **[6]**, which follows Corollary 1.7, shows that the condition "Y has a point-countable base" in Proposition 3.4 cannot be weakened to "Y is first countable".

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