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ON WEYL FRACTIONAL CALCULUS

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ABSTRACT. The Weyl fractional calculus is applied in developing the Laplace transform of $t^q f(t)$, for all values of q. Also, a generalized Taylor's formula in Weyl fractional calculus is established. The results are then used in deriving a certain generating function for the H-function of Fox.

1. Introduction. Let A denote a class of good functions. By a good function f, we mean (Miller [6, p. 82]) a function which is everywhere differentiable any number of times and if it and all of its derivatives are $O(x^{-v})$, for all v as x increases without limit.

We define the Weyl fractional derivatives of a function $g(z)$ as follows: Let $g \in A$, then

$$ {}_zD_\infty^q g(z) = \frac{(-1)^q}{\Gamma(-q)} \int_z^\infty (u-z)^{-q-1} g(u)\,du, \quad \text{for } q < 0. \tag{1}$$

For $q \geqslant 0$,

$$ {}_zD_\infty^q g(z) = \frac{d^r}{dz^r}\left({}_zD_\infty^{q-r} g(z)\right), \tag{2}$$

r being a positive integer such that $r > q$.

We recall the definition of H-function of Fox [3, p. 408] in the form:

$$H(z) = H_{P,Q}^{M,N}\left[z \,\middle|\, \begin{matrix} (a_i, \alpha_i)_{1,P} \\ (b_i, \beta_i)_{1,Q} \end{matrix} \right] = \frac{1}{2\pi\omega}\int_L \theta(s) z^s\,ds, \qquad \omega = \sqrt{-1}, \tag{3}$$

where, for convenience,

$$\theta(s) = \frac{\prod_{i=1}^M \Gamma(b_i - \beta_i s)\prod_{i=1}^N \Gamma(1 - a_i + \alpha_i s)}{\prod_{i=M+1}^Q \Gamma(1 - b_i + \beta_i s)\prod_{i=N+1}^P \Gamma(a_i - \alpha_i s)}, \tag{4}$$

$z \neq 0$, and an empty product is interpreted as unity. The integers M, N, P, Q, are such that $0 \leqslant M \leqslant Q$, $0 \leqslant N \leqslant P$; the coefficients α_i $(i = 1, \ldots, P)$, β_i

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($i = 1, \ldots, Q$), are all assumed to be positive. L is a suitably chosen contour such that all the poles of $\theta(s)$ are simple.

The H-function $H(z)$ is a very general function and has for its particular cases a number of important special functions. For the conditions of existence of the function (3) and its various special cases, the paper of Gupta and Jain [4, pp. 596–600] may be referred to.

2. The Laplace transform of $t^q f(t)$, for arbitrary q.

THEOREM 1. *Let $f(t)$ be such that*

$$g(p) = L[f(t); p] \quad \textit{exists and belongs to } A, \tag{5}$$

where $L[f(t); p]$ denotes the Laplace transform of $f(t)$. Then, for all q,

$$(-1)^q {}_pD_\infty^q g(p) = L[t^q f(t); p]. \tag{6}$$

PROOF. For $q < 0$, we have in view of (1)

$$\begin{aligned}(-1)^q {}_pD_\infty^q g(p) &= \frac{1}{\Gamma(-q)} \int_p^\infty (u-p)^{-q-1} g(u)\, du \\ &= \frac{1}{\Gamma(-q)} \int_p^\infty \left((u-p)^{-q-1} \int_0^\infty e^{-ut} f(t)\, dt\right) du \\ &= \frac{1}{\Gamma(-q)} \int_0^\infty f(t)\left(\int_p^\infty (u-p)^{-q-1} e^{-ut}\, du\right) dt.\end{aligned}$$

Using Erdélyi [2, p. 202, (11)] to evaluate the inner integral on the R.H.S, we get

$$(-1)^q {}_pD_\infty^q g(p) = L[t^q f(t); p], \quad \text{for } q < 0. \tag{7}$$

For $q \geqslant 0$, invoking the definition (2), we can write:

$$(-1)^q {}_pD_\infty^q g(p) = \frac{d^r}{dp^r}\left(\int_0^\infty e^{-pt} t^{q-r} f(t)\, dt\right). \tag{8}$$

Differentiating under the sign of integral, we again find that

$$(-1)^q {}_pD_\infty^q g(p) = L[t^q f(t); p], \quad \text{for } q \geqslant 0. \tag{9}$$

This completes the proof of the theorem.

3. The generalized Taylor's formula. We prove the following theorem which may be regarded as a generalization of the familiar Taylor's formula.

THEOREM 2. *Let*

(i) *a be a real number such that* $0 < a \leqslant 1$,

(ii) *η be an arbitrary complex number and*

(iii) $g(p) \in A$.

Then

$$g(p+t) = \sum_{n=-\infty}^{\infty} \frac{a t^{an+\eta}}{\Gamma(an+\eta+1)}\, {}_pD_\infty^{an+\eta} g(p) \tag{10}$$

is valid for all t on the circle $|t/p| = 1$.

PROOF. Since $g \in A$, we may suppose $g(z) = z^{-\lambda}f(z)$. Assume $f(z) = \sum_{r=0}^{\infty} C_r z^r$.

The L.H.S. of (10) then gives $(p+t)^{-\lambda}\sum_{r=0}^{\infty} C_r(p+t)^r$. Now the R.H.S. of (10) is

$$\sum_{n=-\infty}^{\infty} \frac{at^{an+\eta}}{\Gamma(an+\eta+1)}\, p^{D_\infty^{an+\eta}}\left(p^{-\lambda}\sum_{r=0}^{\infty} C_r p^r\right)$$

$$= \sum_{n=-\infty}^{\infty} \frac{at^{an+\eta}}{\Gamma(an+\eta+1)} \sum_{r=0}^{\infty} C_r \cdot p^{D_\infty^{an+\eta}} p^{-(\lambda-r)}.$$

Since

$$z^{D_\infty^q} z^{-\lambda} = \frac{(-1)^q\Gamma(\lambda+q)}{\Gamma(\lambda)} z^{-\lambda-q}, \tag{11}$$

valid for all q, the R.H.S. of (10) after a little simplification reduces to

$$p^{-\lambda}\sum_{r=0}^{\infty} C_r \cdot p^r \sum_{n=-\infty}^{\infty} a\binom{\lambda - r + an + \eta - 1}{an+\eta}\left(-\frac{t}{p}\right)^{an+\eta}$$

$$= p^{-\lambda}\sum_{r=0}^{\infty} C_r p^r \sum_{n=-\infty}^{\infty} a\binom{-\lambda+r}{an+\eta}\left(\frac{t}{p}\right)^{an+\eta},$$

$$\text{since } (-1)^r\binom{n}{r} = \frac{(-n)r}{r!}.$$

The inner sum can be simplified by making use of the known formula of Osler [7, p. 46, (5.1)], viz.

$$\sum_{n=-\infty}^{\infty} a\binom{p}{an+\eta} t^{an+\eta} = (1+t)^p, \quad \text{for } |t| = 1, \tag{12}$$

to get $\sum_{r=0}^{\infty} C_r(p+t)^{-\lambda+r}$. This completes the proof of Theorem 2.

4. Generating functions. Suppose

$$f(t) = t^{\sigma-1}H_{P,Q}^{M,N}\left[at^h \,\middle|\, \begin{matrix}(a_i,\alpha_i)_{1,P}\\ (b_i,\beta_i)_{1,Q}\end{matrix}\right], \qquad h > 0. \tag{13}$$

Then from (5) and Gupta [5, p. 83],

$$g(p) = p^{-\sigma}H_{P+1,Q}^{M,N+1}\left[ap^{-h} \,\middle|\, \begin{matrix}(1-\sigma,h),(a_i,\alpha_i)_{1,P}\\ (b_i,\beta_i)_{1,Q}\end{matrix}\right], \tag{14}$$

provided that

$$\mathrm{Re}[\sigma + h(b_i/\beta_i)] > 0 \quad (i = 1,\ldots,M), \quad |\arg a| < \tfrac{1}{2}K\pi,$$

where

$$K = \sum_{1}^{M}(\beta_i) - \sum_{M+1}^{Q}(\beta_i) + \sum_{1}^{N}(\alpha_i) - \sum_{N+1}^{P}(\alpha_i) > 0.$$

Also, we have for arbitrary q,

$$z^{D^q_\infty} z^\lambda H^{M,N}_{P,Q}\left[az^{-h}\left|\begin{matrix}(a_i,\alpha_i)_{1,P}\\(b_i,\beta_i)_{1,Q}\end{matrix}\right.\right]$$

$$= (-1)^q z^{\lambda-q} H^{M,N+1}_{P+1,Q+1}\left[az^{-h}\left|\begin{matrix}(1+\lambda-q,h),(a_i,\alpha_i)_{1,P}\\(b_i,\beta_i)_{1,Q},(1+\lambda,h)\end{matrix}\right.\right], \tag{14a}$$

provided that

$$\operatorname{Re}\left[-\lambda + h\tfrac{b_i}{\beta_i}\right] > 0 \quad (i = 1,\dots,M),\ h > 0,\ |\arg a| < \tfrac{1}{2}K\pi,$$

$$\text{where } K = \sum_1^M(\beta_i) - \sum_{M+1}^Q(\beta_i) + \sum_1^N(\alpha_i) - \sum_{N+1}^P(\alpha_i) > 0.$$

(14a) can be established easily by using the contour integral representation (3) and (11).

Substituting $f(t)$ and $g(p)$ from (13) and (14) respectively in the generalized Taylor's formula (10) and using (14a) appropriately, we get

$$(p+t)^{-\sigma} H^{M,N+1}_{P+1,Q}\left[a'(p+t)^{-h}\left|\begin{matrix}(1-\sigma,h),(a_i,\alpha_i)_{1,P}\\(b_i,\beta_i)_{1,Q}\end{matrix}\right.\right]$$

$$= p^{-\sigma}\sum_{n=-\infty}^{\infty}\frac{a(-t/p)^{an+\eta}}{\Gamma(an+\eta+1)}$$

$$\cdot H^{M,N+1}_{P+1,Q}\left[a'p^{-h}\left|\begin{matrix}(1-\sigma-an-\eta,h),(a_i,\alpha_i)_{1,P}\\(b_i,\beta_i)_{1,Q}\end{matrix}\right.\right]. \tag{15}$$

Putting $p = 1$, $\sigma = 1 - a_1$, $h = \alpha_1$ and adjusting the other parameters, we get the generating function

$$(1+t)^{a_1-1} H^{M,N}_{P,Q}\left[z(1+t)^{-\alpha_1}\left|\begin{matrix}(a_i,\alpha_i)_{1,P}\\(b_i,\beta_i)_{1,Q}\end{matrix}\right.\right]$$

$$= \sum_{n=-\infty}^{\infty}\frac{a(-t)^{an+\eta}}{\Gamma(an+\eta+1)} H^{M,N}_{P,Q}\left[z\left|\begin{matrix}(a_1-an-\eta,\alpha_1),(a_i,\alpha_i)_{2,P}\\(b_i,\beta_i)_{1,Q}\end{matrix}\right.\right]. \tag{16}$$

Obviously for $\eta = 0$ and $a = 1$, (16) corresponds to a known result of Anandani [**1**, p. 6, (2.1)].

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